GLOBAL ATTRACTIVITY OF NASH EQUILIBRIA OF A LABOR-MANAGED OLIGOPOLY MODEL

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ABSTRACT. We consider the dynamic model of a labor-managed oligopoly in the form of a system of differential equations. It has been known from our prior work that under natural, verifiable conditions (equivalent to asymptotically constant output of the industry) this system has a ray of equilibria which is locally attracting. Also, every solution which tends to the ray of equilibria, converges to a particular equilibrium on this ray. In the current paper we show that the ray of equilibrium is also globally attracting and give a constructive estimate for the exponential convergence to the equilibrium. We also prove global attractivity in an infinite-dimensional model consisting of an infinite number of competing firms.

CONTENTS

1. INTRODUCTION

The current paper is devoted to the problem of global attractivity of a dynamic model of an economic system called labor-managed oligopoly [9]. Such models have been studied in recent years. The current model has been introduced and examined for the existence and local attractivity of equilibria in [7, 6]. We will summarize previously obtained results in this introduction.

Key words and phrases. labor-managed, oligopoly, global, stability, attractivity.

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1.1. **The description of the model.** First, let us briefly describe the economic model in question. Let us consider an n-firm industry, where all firms are labor-managed. Let us assume the hyperbolic price function:

$$p(s) \;=\; \frac{b}{s}$$

where $s = \sum_{i=1}^{n} x_i$ is the total output of the industry, and linear production functions l_i, and linear labor-independent cost functions c_i:

$$l_i(x_i) \;=\; a_i x_i \qquad \text{and} \qquad c_i(x_i) \;=\; \alpha_i x_i + \beta_i\,,$$

where x_i is the output of firm i ($i = 1, \ldots, n$).

Economic interpretation requires that all parameters b, a_i, α_i, and β_i be positive. The surplus per unit of labor for firm i is given by

$$\begin{aligned}\varphi_i(x_1, \ldots, x_n) \;&=\; \frac{x_i\, p(s) - w\, l_i(x_i) - c_i(x_i)}{l_i(x_i)} \\ &=\; \frac{b}{a_i(x_i + Q_i)} - w - \frac{\alpha_i}{a_i} - \frac{\beta_i}{a_i\, x_i}\end{aligned} \tag{1.1}$$

where $Q_i = \sum_{l \neq i} x_l$ is the output of the rest of the industry, and w is the competitive wage rate.

We note that in a labor-managed oligopoly, by definition the objective is to maximize the profit per unit of labor, and thus φ_i is the payoff function, in the language of game theory.

1.2. **The marginal profit.** This is defined as the rate of change of payoff resulting from only changing x_i and keeping the output of the rest of the industry fixed. In general, the marginal profit of firm i for our class of models is:

$$\left.\frac{\partial}{\partial x_i}\right|_{Q_i=\text{const}} \left(\frac{x_i\, p(x_i + Q_i) - w\, l_i(x_i) - c_i(x_i)}{l_i(x_i)}\right) \tag{1.2}$$

and for our particular model it is:

$$\left.\frac{\partial}{\partial x_i}\right|_{Q_i=\text{const}} \left(\frac{b}{a_i(x_i + Q_i)} - w - \frac{\alpha_i}{a_i} - \frac{\beta_i}{a_i\, x_i}\right) = -\frac{b}{a_i\, s^2} + \frac{\beta_i}{a_i\, x_i^2} \tag{1.3}$$

1.3. **The existence of positive equilibria.** The model of labor-managed oligopoly just described shall be regarded as an n-person game where the set of strategies for each firm is the interval $X_i = [0, \infty)$ and the payoff function of firm i is φ_i. Thus, for each firm i and $Q_i > 0$, the best response is obtained as

$$x_i(Q_i) \;=\; \text{argmax}_{x_i \geq 0} \left\{\frac{b}{a_i(x_i + Q_i)} - w - \frac{\alpha_i}{a_i} - \frac{\beta_i}{a_i\, x_i}\right\}. \tag{1.4}$$

For completeness, we state and prove the result from [7] concerning the existence of the above optimum.

Theorem 1.1. *An interior optimum (i.e. satisfying $x_i > 0$ for all i) exists for conditions (1.4) iff $\beta_i < b$. If this condition is satisfied then all positive equilibria of the labor-managed oligopoly problem exist if and only if*

$$\sum_{i=1}^{n} \sqrt{\beta_i} \;=\; \sqrt{b}\,.$$

If this condition is satisfied then

$$\hat{x}_i = \frac{\sqrt{\beta_i}}{\sqrt{b}}\,\hat{s}, \quad i = 1, \ldots, n, \tag{1.5}$$

is an equilibrium for every positive $\hat{s}$. Moreover, if we require that the payoff be positive for every firm then $\hat{s}$ must satisfy the relation

$$\hat{s} < \min_i \left\{ \frac{\sqrt{b}\,(\sqrt{b}-\sqrt{\beta_i})}{a_i\,w+\alpha_i} \right\}. \tag{1.6}$$

All positive equilibria can be obtained in this way.

Proof. Assuming an interior optimum, the first order conditions are:

$$-\frac{b}{a_i\,(x_i+Q_i)^2}+\frac{\beta_i}{a_i\,x_i^2} = 0\,.$$

This system of equations has the following solution:

$$x_i = \frac{\sqrt{\beta_i}}{\sqrt{b}-\sqrt{\beta_i}}\,Q_i, \qquad (i=1,2,\ldots,n)\,. \tag{1.7}$$

In order to ensure that $x_i > 0$, we have to assume that $\beta_i < b$. The second order conditions are always satisfied since at the optimum

$$\frac{2b}{a_i\,(x_i+Q_i)^3}-\frac{2\beta_i}{a_i\,x_i^3} = \frac{2\beta_i}{a_i\,x_i^3}\left(\sqrt{\frac{\beta_i}{b}}-1\right) < 0\,.$$

From equation (1.7) we have

$$s = x_i+Q_i = x_i\left(1+\frac{\sqrt{b}-\sqrt{\beta_i}}{\sqrt{\beta_i}}\right) = x_i\,\frac{\sqrt{b}}{\sqrt{\beta_i}},$$

and finally,

$$1 = \frac{\sum_{i=1}^n x_i}{s} = \frac{\sum_{i=1}^n \sqrt{\beta_i}}{\sqrt{b}}\,.$$

If $\hat{s}$ denotes this optimal value of s then the equations (1.5) is satisfied.

The payoff of firm i at any equilibrium is

$$\varphi_i(\hat{x}_1,\ldots,\hat{x}_n) = \frac{b}{a_i\,\hat{s}}-w-\frac{\alpha_i}{a_i}-\frac{\beta_i}{a_i\,\hat{x}_i} = \frac{b}{a_i\,\hat{s}}\left(1-\frac{\sqrt{\beta_i}}{\sqrt{b}}\right)-w-\frac{\alpha_i}{a_i}$$

which is positive for all i if and only if $\hat{s}$ is sufficiently small, more precisely, if the inequalities (1.6) hold. □

1.4. **The dynamic model and attractivity.** The dynamic model of a labor-managed oligopoly is constructed by assuming continuous time scale and that each firm adjusts its output continuously and proportionally to its marginal profit.

The resulting dynamic model is:

$$\dot{x}_i = k_i\left(-\frac{b}{a_i\,s^2}+\frac{\beta_i}{a_i\,x_i^2}\right) \qquad i=1,2,\ldots,n \tag{1.8}$$

where $k_i > 0$ is a constant specified for each i. We also assume that $x_i > 0$ for all i, which is consistent with the physical interpretation of x_i as output of the i-th firm.

In [7] the existence and local attractivity of the equilibria of this system was shown. The main result of this paper can be stated as follows:

Theorem 1.2. *Let us assume that for $i = 1, 2, \ldots, n$ we have $\beta_i < b$ and moreover, $\sum_{i=1}^{n} \sqrt{\beta_i} = \sqrt{b}$. Then in dynamical system (1.8) all equilibria form an open ray starting from the origin, given by the parametric equation*

$$x_i = \frac{\sqrt{\beta_i}}{\sqrt{b}} s, \quad s > 0, \ i = 1, \ldots, n.$$

This is a strongly attracting invariant set, i.e. every solution $x_i(t)$, $i = 1, 2, \ldots, n$ to the model will approach one of the equilibria on this ray with an exponential rate of convergence, as $t \to \infty$.

The above theorem was proved in [7] using methods of dynamical systems theory concerning the existence of stable foliations for normally hyperbolic manifolds. This theory is rather complicated and will not be presented in the current paper, but the interested reader may find an accessible exposition in [7]. Invariant manifold theory will not be required to understand the results of the current paper.

The assumption $\sum_{i=1}^{n} \sqrt{\beta_i} = \sqrt{b}$ used in Theorem 1.2 is a limitation of the current paper, but it is necessary to remain in the realm of simple-minded attractivity theory, where attractivity is understood as convergence to an equilibrium. This assumption will be dropped in our next paper, which investigates the cases $\sum_{i=1}^{n} \sqrt{\beta_i} > \sqrt{b}$ and $\sum_{i=1}^{n} \sqrt{\beta_i} < \sqrt{b}$. It proves that in the first case the total output of the industry grows to infinity, while the ratio x_i/s (i.e. the share of the total output of the industry for firm i) goes to a definite limit. We will refer to this situation as *dynamic equilibrium*. In the case $\sum_{i=1}^{n} \sqrt{\beta_i} < \sqrt{b}$ the total output of the industry shrinks to 0 and multiple dynamic equilibria are possible. A detailed development of the theory of dynamic equilibria is the main subject of [10].

2. Global attractivity of the ray of equilibria for labor-managed oligopoly

The main result of this section is the following global version of Theorem 1.2.

Theorem 2.1. *Let us assume that for $i = 1, 2, \ldots, n$ we have $\beta_i > 0$, $b > 0$, $\beta_i < b$ and*

$$\sum_{i=1}^{n} \sqrt{\beta_i} = \sqrt{b}.$$

Then every solution $x_i(t)$, $i = 1, 2, \ldots, n$ of the dynamical system (1.8) is defined on $[0, \infty)$ and converges to one of the points on the ray

$$x_i = \frac{\sqrt{\beta_i}}{\sqrt{b}} s, \quad s > 0, \ i = 1, \ldots, n$$

as $t \to \infty$. Thus, the ray is a globally strongly attracting set.

Proof. First we notice that the local solution to the system exists for any initial condition $\mathbf{x}_0 > \mathbf{0}$ because the right hand side of the system is C^1.

The first step in the proof is a change of coordinates. The new coordinates are given as follows:

$$y_i = \frac{\sqrt{b}}{\sqrt{\beta_i}} x_i.$$

In these new coordinates the model can be written as

$$\dot{y}_i = \frac{\sqrt{b}}{\sqrt{\beta_i}} \frac{k_i b}{a_i} \left(-\frac{1}{s^2} + \frac{1}{y_i^2} \right)$$

where $s = \sum_{i=1}^{n} \frac{\sqrt{\beta_i}}{\sqrt{b}} y_i$. Let us introduce the following notation:

$$\begin{aligned} \lambda_i &= \frac{\sqrt{b}}{\sqrt{\beta_i}} \frac{k_i b}{a_i}, \\ \gamma_i &= \frac{\sqrt{\beta_i}}{\sqrt{b}}. \end{aligned}$$

The main point is that s is a *convex* combination of the coordinates y_i, i.e. $\sum_{i=1}^{n} \gamma_i = 1$. The new system of differential equations can be written in a simplified form:

$$\begin{aligned} \dot{y}_i &= \lambda_i \left(-\frac{1}{s^2} + \frac{1}{y_i^2} \right), \\ s &= \sum_{i=1}^{n} \gamma_i y_i. \end{aligned} \tag{2.1}$$

It is also quite clear that the equilibria of this system are the points where y_i are all identical.

Let us introduce two functions, $M, m : \mathbb{R}^n \to \mathbb{R}_+$ via the following definitions:

$$\begin{aligned} M(\mathbf{y}) &= \max_{1 \le i \le n} y_i \\ m(\mathbf{y}) &= \min_{1 \le i \le n} y_i \end{aligned}$$

We claim that M is a decreasing function of time and m is an increasing function of time along any trajectory. First we show that M decreases along the solutions of our model.

For any fixed $\mathbf{y} \in \mathbb{R}^n$, let

$$\mathcal{I}(\mathbf{y}) = \{ j \,:\, 1 \le j \le n,\, y_j = M(\mathbf{y}) \} \tag{2.2}$$

i.e. this is the set of those indices that the maximum is attained. We claim that $\dot{y}_j(t) \le 0$ for every $j \in \mathcal{I}(\mathbf{y}(t))$. Indeed,

$$s = \sum_{i=1}^{n} \gamma_i y_i \le \sum_{i=1}^{n} \gamma_i M(\mathbf{y}) = M(\mathbf{y}) = y_j$$

Moreover, the equality holds only if $\gamma_j = 0$ for all $j \notin \mathcal{I}(\mathbf{y})$. As we assume that all γ_j are positive, this happens only when $\mathcal{I}(\mathbf{y})$ contains all j in the range $1 \le j \le n$, i.e. when all y_j are equal and $\mathbf{y}$ is in the equilibrium set.

Hence, if $y_j = y_j(t)$ is a solution of our model defined on some interval $[0, a)$ then for every $t \in [0, a)$ we have:

$$\dot{y}_j = \lambda_i \left(-\frac{1}{s^2} + \frac{1}{y_j^2} \right) \le \lambda_i \left(-\frac{1}{y_j^2} + \frac{1}{y_j^2} \right) = 0 \tag{2.3}$$

for $j \in \mathcal{I}(\mathbf{y})$. Moreover, the equality holds for all j only when $\mathbf{y}(t)$ is an equilibrium.

We observe that the set $\mathcal{I}(\mathbf{y})$ is an *upper semi-continuous*, set-valued function of $\mathbf{y}$.

We recall that a set-valued function $f : X \to 2^Y$ from a topological (or metric) space X to the power set 2^Y of another topological (or metric) space Y is called upper semi-continuous if for any $\mathbf{x}_0 \in X$, and any open set $O \subset Y$ containing $f(\mathbf{x}_0)$, then there exists an open set $W \subset X$ containing $\mathbf{x}_0$ such that for every $\mathbf{x} \in W$ we have $f(\mathbf{x}) \subseteq O$. If Y is a discrete topological space, i.e. every subset is open, then we can only consider $O = f(\mathbf{x}_0)$ in the above definition, and thus obtain the following abbreviated condition of upper semi-continuity: for every $\mathbf{x}_0 \in X$ there is an open set $W \subseteq X$ containing $\mathbf{x}_0$ such that for every $\mathbf{x} \in W$ we have $f(\mathbf{x}) \subseteq f(\mathbf{x_0})$. For a nice, comprehensive introduction to general topology including upper semi-continuity, the reader may consult [3], but most advanced textbooks on general topology will cover this subject.

Upper semi-continuity means that the set $\mathcal{I}(\mathbf{y})$ can only decrease in a small neighborhood of $\mathbf{y}$, and never increase. More precisely, let

$$\epsilon = \epsilon(\mathbf{y}) = \min_{j \notin \mathcal{I}(\mathbf{y})} \{M(\mathbf{y}) - y_j\}. \tag{2.4}$$

Thus, ϵ is the gap between all maximal values of y_j and all remaining values of y_j. We note that ϵ is well defined only when the minimum is not over an empty set, i.e. when y is not in the equilibrium set of our model.

We claim that if $z \in \mathbb{R}^n$ and $\|\mathbf{z} - \mathbf{y}\|_\infty < \epsilon/2$ then

$$\mathcal{I}(\mathbf{z}) \subseteq \mathcal{I}(\mathbf{y}).$$

Here $\|\mathbf{z} - \mathbf{y}\|_\infty$ denotes the *sup-norm* of the vector $\mathbf{z} - \mathbf{y}$, i.e. $\max_{j:1\le j\le n} |z_j - y_j|$. Indeed, M is a *weak contraction*, i.e. for all $\mathbf{w}_1$, $\mathbf{w}_2$ we have

$$|M(\mathbf{w}_1) - M(\mathbf{w}_2)| \le \|\mathbf{w}_1 - \mathbf{w}_2\|_\infty.$$

Thus, $M(\mathbf{z}) \ge M(\mathbf{y}) - \epsilon/2$. Also, every individual coordinate is a Lipschitz function of $\mathbf{y}$ with constant 1, i.e. $|z_j - y_j| \le \|\mathbf{z} - \mathbf{y}\| < \epsilon/2$. Thus, in particular $z_j < y_j + \epsilon/2$. Hence, if $j \notin \mathcal{I}(\mathbf{y})$ then $z_j < y_j + \epsilon/2 \le M(\mathbf{y}) - \epsilon + \epsilon/2 \le (M(\mathbf{z}) + \epsilon/2) - \epsilon + \epsilon/2 = M(\mathbf{z})$. Thus, $j \notin \mathcal{I}(\mathbf{z})$. The upper semi-continuity of the function $\mathcal{I}(\mathbf{y})$ has been shown.

Thus if $\mathbf{y}(t)$ is a solution of our system defined on some interval $[0, a)$, and $\mathbf{y}(t)$ is not an equilibrium, then the function $g(t) = M(\mathbf{y}(t))$ is strictly decreasing on the interval $[0, a)$. We apply a version of a classical argument in real analysis, which is a proof by contradiction.

Let us suppose that $g(t)$ is not strictly decreasing on its domain $[0, a)$. Let us define

$$t_0 = \sup\{\tau \in [0, a) \,:\, g(t) \text{ is strictly decreasing on } [0, \tau)\}.$$

Clearly, $t_0 < a$. By continuity of the solution $\mathbf{y}(t)$, there exists $\delta > 0$ such that for every $t \in [t_0, t_0 + \delta)$

$$\|\mathbf{y}(t) - \mathbf{y}(t_0)\|_\infty < \epsilon/2$$

where $\epsilon = \epsilon(\mathbf{y}(\mathbf{t_0}))$ (cf. (2.4)), and in particular $\mathcal{I}(\mathbf{y}(t)) \subseteq \mathcal{I}(\mathbf{y}(t_0))$. There is also a number $\delta_1 \in (0, \delta)$ such that for all $t \in [t_0, t_0 + \delta_1)$ and $j \in \mathcal{I}(\mathbf{y}(t_0))$ we have $y_j(t) < y_j(t_0)$. Hence,

$$\begin{aligned} g(t) &= M(\mathbf{y}(t)) = \max_{1\le j\le n} y_j(t) \\ &= \max_{j \in \mathcal{I}(y(t))} y_j(t) \le \max_{j \in \mathcal{I}(y(t_0))} y_j(t) \\ &< \max_{j \in \mathcal{I}(y(t_0))} y_j(t_0) = g(t_0). \end{aligned}$$

We note that we used the upper semi-continuity of $\mathcal{I}(\mathbf{y})$ in the above calculation: we needed $\mathcal{I}(\mathbf{y}(\mathbf{t})) \subseteq \mathcal{I}(\mathbf{y}(t_0))$. The above contradicts the definition of t_0, as the function would be also strictly decreasing on $[0, t_0 + \delta_1)$. Hence, the function $g(t)$ is strictly decreasing on its entire domain $[0, a)$.

In a similar fashion, we show that m strictly increases along integral curves which do not start at an equilibrium. Of course, both M and m remain constant along solutions starting at an equilibrium.

The above information is sufficient to show that every solution $\mathbf{y}(t)$ can be extended to $t \in [0, \infty)$, i.e. the system is *forward complete* (see Appendix C). In fact, every solution $\mathbf{y}(t)$ lies in the set

$$\{\mathbf{y} \,:\, m(\mathbf{y}(0)) \le m(\mathbf{y}) \le M(\mathbf{y}) \le M(\mathbf{y}(0))\},$$

which is a Cartesian product of closed intervals, and thus is a compact set in the Euclidean space. We observe that Lemma C.3 of Appendix C implies that every solution $\mathbf{y}(t)$ is defined on $[0,\infty)$, i.e. the system is forward complete.

At this point of the proof we have two options. One is to apply a version of LaSalle's Invariance Principle (see Appendix E) or a direct argument.

It remains to be shown that the ray of equilibria is a strongly attracting set, i.e. for every solution $\mathbf{y}(t)$ defined for $t \in [0,\infty)$ the limit $\lim_{t\to\infty} \mathbf{y}(t)$ exists and belongs to the ray of equilibria.

First, let us explain how our result follows from Appendix E. Let S be the ray of equilibria and let $V(\mathbf{y}) = M(\mathbf{y})$. The set S is trivially forward invariant. The function V is strictly decreasing and $V|_S$ is injective. Also, we have just shown that every trajectory remains in a compact set. Thus the strong attractivity of S follows immediately from Theorem E.1.

As an alternative, without any concern for generality, we present a direct proof of the same conclusion, the global attractivity of the ray of equilibria.

Both limits, $\overline{M} = \lim_{t\to\infty} M(\mathbf{y}(t))$ and $\overline{m} = \lim_{t\to\infty} m(\mathbf{y}(t))$, exist as limits of monotonic and bounded functions. We claim that they are equal. The proof of this claim is by contradiction. Let us suppose that $\overline{M} \neq \overline{m}$. For every $\epsilon > 0$ the set

$$U_\epsilon = \{\mathbf{y} \in \mathbb{R}^n \,:\, \overline{m} - \epsilon < m(\mathbf{y}) \text{ and } M(\mathbf{y}) < \overline{M} + \epsilon\}.$$

is open and bounded. Moreover, by definition, for sufficiently large t the point $\mathbf{y}(t)$ is in U_ϵ. Thus any limit point of the solution $\mathbf{y}(t)$ lies in the intersection of all U_ϵ, which is the compact set

$$B = \{\mathbf{y} \in \mathbb{R}^n \,:\, \overline{m} \le m(\mathbf{y}) \text{ and } M(\mathbf{y}) \le \overline{M}\}.$$

Let $\mathbf{z}_0 \in B$ be any limit point of the solution $\mathbf{y}(t)$. Such points exist by compactness of B. Thus, there exists a sequence of times $t_n \to \infty$ such that $\mathbf{y}(t_n) \to \mathbf{z}_0$. In view of continuity of the functions M and m, we have $M(\mathbf{z}_0) = \overline{M}$ and $m(\mathbf{z}_0) = \overline{m}$. We assumed that $\overline{M} \neq \bar{m}$, and thus $\mathbf{z}_0$ is not an equilibrium. Let $\mathbf{z}(t)$ be the solution with initial condition $\mathbf{z}(0) = \mathbf{z}_0$. The function $M(\mathbf{z}(t))$ is decreasing, and thus there exists $\delta > 0$ and $T > 0$ such that $M(\mathbf{z}(T)) < M(\mathbf{z}_0) - \delta$. Continuous dependence of solutions on initial conditions implies that there is a neighborhood V of $\mathbf{z}_0$ such that for any $\mathbf{w}_0 \in V$ we have $M(\mathbf{w}(T)) < M(\mathbf{w}_0) - \delta$ and $m(\mathbf{w}(T)) > m(\mathbf{w}_0) + \delta$, where $\mathbf{w}(t)$ is the solution to the initial value problem with initial condition $\mathbf{w}(0) = \mathbf{w}_0$. From the monotonicity of M and m along any non-equilibrium trajectory we also conclude that for all $t \ge T$ we have $M(\mathbf{w}(t)) < M(\mathbf{w}_0) - \delta$ and $m(\mathbf{w}(t)) > m(\mathbf{w}_0) + \delta$. In particular, for every natural number n such that $\mathbf{y}(t_n) \in V$, there is a natural number k such that $t_{n+k} - t_n > T$, which implies $M(\mathbf{y}(t_{n+k})) < M(\mathbf{y}(t_n)) - \delta$ and $m(\mathbf{y}(t_{n+k})) > m(\mathbf{y}(t_n)) + \delta$. This contradicts the convergence of the sequences $M(\mathbf{y}(t_n))$ and $m(\mathbf{y}(t_n))$. Thus, we have shown that $\overline{M} = \overline{m}$, which implies that B consists exactly of one point $\mathbf{z}_0$, which is automatically an equilibrium. Therefore, $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{z}_0$.

□

3. Exponential convergence to the equilibrium

3.1. **The framework for exponential convergence.** In this section we will demonstrate that the convergence to the equilibrium in Theorem 2.1 is exponential.

This local result has been shown in [7] by an analysis of the spectrum of the linearized system at the equilibrium. The current result is satisfying in other respects. For instance, it provides a constructive bound on the exponent, given explicitly in terms of the parameters of the system.

The main idea of this section is to introduce the function

$$V(\mathbf{y}) = M(\mathbf{y}) - m(\mathbf{y})$$

and prove an estimate $\dot{V} \leq -\alpha V$ valid uniformly on the entire space. This implies that $V(t) \leq V(0)e^{-\alpha t}$ and thus $V(t) \to 0$ exponentially.

The technical difficulty is that V is not differentiable. Thus, we must make weaker statements which will imply the same conclusion.

The main observation leading to the proof of the exponential convergence is in the following lemma:

Lemma 3.1. *Let*

$$\alpha = \alpha(\mathbf{y}) = \frac{2\min_i \lambda_i}{M(\mathbf{y})^3}.$$

For every $\mathbf{y} \in \mathbb{R}^n$ and every pair of indices i and j such that $y_i \geq s \geq y_j$ we have:

$$\dot{y}_i - \dot{y}_j \leq -\alpha(y_i - y_j).$$

Proof. For each i such that $y_i \geq s$ we have the following estimate of $\dot{y}_i$:

$$\begin{aligned} \dot{y}_i &= \lambda_i\left(-\frac{1}{s^2} + \frac{1}{y_i^2}\right) = \lambda_i\left(-\frac{1}{s^2} + \frac{1}{(s+(y_i-s))^2}\right) \\ &= -\frac{2\lambda_i}{(s+\theta(y_i-s))^3}(y_i-s) \leq -\frac{2\lambda_i}{M(\mathbf{y})^3}(y_i-s). \end{aligned}$$

(We note that we used the Mean Value Theorem, and thus $\theta \in [0,1]$.)

Similarly, for each j such that $y_j \leq s$ we have

$$\begin{aligned} \dot{y}_j &= \lambda_j\left(-\frac{1}{s^2} + \frac{1}{y_j^2}\right) = \lambda_j\left(-\frac{1}{s^2} + \frac{1}{(s-(s-y_j))^2}\right) \\ &= \frac{2\lambda_j}{(s-\theta(s-y_j))^3}(s-y_j) \geq \frac{2\lambda_j}{M(\mathbf{y})^3}(s-y_j). \end{aligned}$$

Thus, we have the following two inequalities:

$$\begin{aligned} \dot{y}_i &\leq -\alpha(y_i - s) \qquad \text{as } y_i \geq s, \\ \dot{y}_j &\geq \alpha(s - y_j) \qquad \text{as } y_j \leq s. \end{aligned}$$

Together, they yield our lemma, as

$$\dot{y}_i - \dot{y}_j \leq -\alpha(y_i - s) - \alpha(s - y_j) = -\alpha(y_i - y_j).$$

□

3.2. **Support theorems from real analysis.** Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the extended line.

In order to handle non-differentiability of V we introduce the *right-upper derivative*. In the current section we develop a calculus of right-upper derivatives to a somewhat greater extent than required by our specific model. Hence, the reader may decide to skip this section upon the first reading, or just become familiar with the next definition and Lemmas 3.9 and 3.10, which are the only results needed in the remainder of the paper. The notion of right-upper derivative is only used in Lemma 3.11, and the conclusions of Lemmas 3.9 and 3.10 are only used in the proof of Theorem 3.13.

Definition 3.1. *Let $u : [p,q) \to \overline{\mathbb{R}}$. Let $U = u^{-1}(\mathbb{R})$. We define two functions, $D_\delta^\dagger u : U \to \overline{\mathbb{R}}$ (*the δ-right-upper derivative*) and $D^\dagger u : U \to \overline{\mathbb{R}}$ (*the right-upper derivative*), as follows:*
(a) For every δ where $0 < \delta < q - p$, the δ-right-upper derivative *of u at a point $t \in U$ is defined by:*

$$(D_\delta^\dagger u)(t) = \sup_{t' \in (t, t+\delta)} \frac{u(t') - u(t)}{t' - t}.$$

We note that $D^\dagger_\delta u(t)$ is decreasing as function of δ.
(b) The right-upper derivative *of u at $t \in [p,q)$ is defined by:*

$$(D^\dagger u)(t) = \lim_{\delta \to 0} D^\dagger_\delta u(t) = \inf_{\delta > 0} D^\dagger_\delta u(t)$$

We will write $D^\dagger_\delta u(t)$ and $D^\dagger u(t)$ in place of the more precise notation $(D^\dagger_\delta u)(t)$ and $(D^\dagger u)(t)$.

We note that $D^\dagger_\delta u : [p, q-\delta] \to \overline{\mathbb{R}}$ and $D^\dagger u : [p,q) \to \overline{\mathbb{R}}$ are well-defined for every function $u : [p,q) \to \mathbb{R}$. In what follows, we will assume the customary extension of arithmetical operations and order relations to the infinite values. The only undefined operations are $\infty - \infty$ and $0 \cdot \infty$ and their equivalents.

We will need some analogues of theorems from differential calculus:

Lemma 3.2. *Let $u, v : [p,q) \to \overline{\mathbb{R}}$ be two arbitrary functions, $U = u^{-1}(\mathbb{R})$, $V = v^{-1}(\mathbb{R})$ be the two sets where these functions assume finite values, and let $\theta > 0$ be an arbitrary positive constant. Let $\delta > 0$.*

1. *For every $t \in U$: $D^\dagger_\delta(\theta u)(t) = \theta D^\dagger_\delta u(t)$ (positive homogeneity);*
2. *For every $t \in U \cap V$: $D^\dagger_\delta(u+v)(t) \le D^\dagger_\delta u(t) + D^\dagger_\delta v(t)$ (subadditivity);*
3. *If in addition $u, v \ge 0$, and v is increasing, then for every $t \in U \cap V$:*

$$D^\dagger_\delta(uv)(t) \le \bar{u}(t) D^\dagger_\delta v(t) + v(t) D^\dagger_\delta u(t)$$

where $\bar{u}$ is an decreasing function defined by $\bar{u}(t) = \sup_{t' \in [t,q)} u(t')$. In particular, if u is decreasing, we have:

$$D^\dagger_\delta(uv)(t) \le u(t) D^\dagger_\delta v(t) + v(t) D^\dagger_\delta u(t)$$

(subadditive product rule);

4. *The above properties hold for $D^\dagger_\delta$ replaced with $D^\dagger$, for every $t \in U$ or $t \in U \cap V$.*
5. *If $u, v \ge 0$, at least one of u and v is right-continuous at $t \in U \cap V$, then*

$$D^\dagger(uv)(t) \le u(t) D^\dagger v(t) + v(t) D^\dagger u(t)$$

(subadditive product rule);

6. *If u is differentiable at $t \in U$ in the usual sense then $D^\dagger u(t) = u'(t)$.*

Proof. (1)

$$D^\dagger_\delta(\theta u)(t) \quad = \sup_{t' \in (t, t+\delta)} \frac{\theta u(t') - \theta u(t)}{t' - t} = \theta \sup_{t' \in (t, t+\delta)} \frac{u(t') - u(t)}{t' - t} = \theta D^\dagger_\delta u(t)$$

(2)

$$\begin{aligned} D^\dagger_\delta(u+v)(t) &= \sup_{t' \in (t,t+\delta)} \frac{u(t') + v(t') - u(t) - v(t)}{t' - t} \\ &\le \Big(\sup_{t' \in (t,t+\delta)} \frac{u(t') - u(t)}{t' - t} + \sup_{t' \in (t,t+\delta)} \frac{v(t') - v(t)}{t' - t} \Big) \\ &= D^\dagger_\delta u(t) + D^\dagger_\delta v(t) \end{aligned}$$

(3)

$$\begin{aligned}
D_\delta^\dagger(uv)(t) &= \sup_{t'\in(t,t+\delta)} \frac{u(t')v(t')-u(t)v(t)}{t'-t} \\
&= \sup_{t'\in(t,t+\delta)} \frac{u(t')v(t')-u(t')v(t)+u(t')v(t)-u(t)v(t)}{t'-t} \\
&= \sup_{t'\in(t,t+\delta)} \left(u(t')\frac{v(t')-v(t)}{t'-t}+v(t)\frac{u(t')-u(t)}{t'-t}\right) \\
&\le \sup_{t'\in(t,t+\delta)} \left(u(t')\frac{v(t')-v(t)}{t'-t}\right)+v(t)\sup_{t'\in(t,t+\delta)}\frac{u(t')-u(t)}{t'-t} \\
&\le \sup_{t'\in(t,t+\delta)} \left(\bar{u}(t)\frac{v(t')-v(t)}{t'-t}\right)+v(t)D_\delta^\dagger u(t) \\
&= \bar{u}(t)D_\delta^\dagger v(t)+v(t)D_\delta^\dagger u(t).
\end{aligned}$$

(4) Trivial.
(5) Without a loss of generality, we assume that u is right-continuous at t. From (3), for $\forall\delta>0$

$$D_\delta^\dagger(uv)(t) \le \sup_{t'\in(t,t+\delta)} \left(u(t')\frac{v(t')-v(t)}{t'-t}\right)+v(t)\sup_{t'\in(t,t+\delta)}\frac{u(t')-u(t)}{t'-t}.$$

If $D^\dagger v(t)<0$, then $\exists\delta_0>0$ such that $D_\delta^\dagger v(t)<0$ whenever $0<\delta\le\delta_0$. Thus

$$\begin{aligned}
D_\delta^\dagger(uv)(t) &\le \inf_{t'\in(t,t+\delta)} u(t')\sup_{t'\in(t,t+\delta)}\frac{v(t')-v(t)}{t'-t}+v(t)D_\delta^\dagger u(t) \\
&= \inf_{t'\in(t,t+\delta)} u(t')D_\delta^\dagger v(t)+v(t)D_\delta^\dagger u(t).
\end{aligned}$$

Letting $\delta\to 0$ and with u being right-continuous at t, we have

$$D^\dagger(uv)(t) \le u(t)D^\dagger v(t)+v(t)D^\dagger u(t).$$

If $D^\dagger v(t)\ge 0$, then $D_\delta^\dagger v(t)\ge 0$ for $\forall\delta>0$. There are two cases. If $\exists\delta_0>0$ such that $D_{\delta_0}^\dagger v(t)=0$, then $D_\delta^\dagger v(t)=0$ for $\forall 0<\delta\le\delta_0$, thus

$$\begin{aligned}
D_\delta^\dagger(uv)(t) &\le \inf_{t'\in(t,t+\delta)} u(t')\sup_{t'\in(t,t+\delta)}\frac{v(t')-v(t)}{t'-t}+v(t)D_\delta^\dagger u(t) \\
&= \inf_{t'\in(t,t+\delta)} u(t')D_\delta^\dagger v(t)+v(t)D_\delta^\dagger u(t).
\end{aligned}$$

Otherwise, $D_\delta^\dagger v(t)>0$ for $\forall\delta>0$. Thus

$$\begin{aligned}
D_\delta^\dagger(uv)(t) &\le \sup_{t'\in(t,t+\delta)} u(t')\sup_{t'\in(t,t+\delta)}\frac{v(t')-v(t)}{t'-t}+v(t)D_\delta^\dagger u(t) \\
&= \sup_{t'\in(t,t+\delta)} u(t')D_\delta^\dagger v(t)+v(t)D_\delta^\dagger u(t).
\end{aligned}$$

Letting $\delta\to 0$ in these two cases and with u being right-continuous at t, we have

$$D^\dagger(uv)(t) \le u(t)D^\dagger v(t)+v(t)D^\dagger u(t).$$

(6) Trivial. $\square$

The reason for this definition is the following property which is lacking for the ordinary differentiation:

Theorem 3.3. *Let $u_l : [p,q) \to \overline{\mathbb{R}}$, $l \in \mathcal{L}$, be any family of functions (finite, countable or uncountable) and let $u = \sup_{l\in\mathcal{L}} u_l$. Let $0 < \delta < q-p$. For all $t \in [p, q-\delta]$ such that $u(t) \in \mathbb{R}$, we have:*

$$D^{\dagger}_{\delta} u(t) \le \sup_{l\in\mathcal{L}} D^{\dagger}_{\delta} u_l(t).$$

Proof. For any two families of numbers $(a_l)_{l\in\mathcal{L}}$ and $(b_l)_{l\in\mathcal{L}}$ we have:

$$\sup_{l\in\mathcal{L}} a_l - \sup_{l\in\mathcal{L}} b_l \le \sup_{l\in\mathcal{L}} (a_l - b_l)\,.$$

Therefore, for any fixed $t, t' \in [p,q)$, and $t < t'$:

$$\frac{\sup_{l\in\mathcal{L}} u_l(t') - \sup_{l\in\mathcal{L}} u_l(t)}{t'-t} \le \sup_{l\in\mathcal{L}} \frac{u_l(t') - u_l(t)}{t'-t}.$$

Thus,

$$\frac{u(t') - u(t)}{t'-t} \le \sup_{l\in\mathcal{L}} \frac{u_l(t') - u_l(t)}{t'-t}.$$

Hence, for any fixed $\delta > 0$:

$$\sup_{t'\in(t,t+\delta)} \frac{u(t') - u(t)}{t'-t} \le \sup_{t'\in(t,t+\delta)} \sup_{l\in\mathcal{L}} \frac{u_l(t') - u_l(t)}{t'-t}.$$

The order of the suprema can be interchanged, and thus

$$\sup_{t'\in(t,t+\delta)} \frac{u(t') - u(t)}{t'-t} \le \sup_{l\in\mathcal{L}} \sup_{t'\in(t,t+\delta)} \frac{u_l(t') - u_l(t)}{t'-t},$$

which means $D^{\dagger}_{\delta} u(t) \le \sup_{l\in\mathcal{L}} D^{\dagger}_{\delta} u_l(t)$. □

Corollary 3.4. (1)

$$D^{\dagger} u(t) \le \lim_{\delta\to 0} \sup_{l\in\mathcal{L}} D^{\dagger}_{\delta} u_l(t)\,.$$

(2) *If $D^{\dagger}_{\delta} u_l(t)$ converge to $D^{\dagger} u_l(t)$ uniformly in $l \in \mathcal{L}$ as $\delta \to 0$, then*

$$D^{\dagger} u(t) \le \sup_{l\in\mathcal{L}} D^{\dagger} u_l(t)\,.$$

Remark 3.5. The uniform convergence condition in part (2) of Corollary 3.4 is necessary in consideration of the following example.

We let $\mathcal{L} = \{1, 2, ...\}$ and define u_l on $[0,\infty)$ via

$$u_l(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1/l, \\ t - \dfrac{1}{l} & \text{if } 1/l < t < \infty. \end{cases}$$

Then $D^{\dagger} u_l(0) = 0$ for all $l \in \mathcal{L}$. On the other hand, $\sup_{l\in\mathcal{L}} u_l(t) = t$ and thus $D^{\dagger}(\sup_{l\in\mathcal{L}} u_l)(0) = 1$. Therefore, $\sup_{l\in\mathcal{L}} D^{\dagger} u_l(0) = 0 < 1 = D^{\dagger}(\sup_{l\in\mathcal{L}} u_l)(0)$.

Let $\mathcal{F} = \mathcal{F}(\mathcal{L})$ be the family of all finite subsets of $\mathcal{L}$. The set $\mathcal{F}$ is ordered by inclusion, and the pair $(\mathcal{F}, \supseteq)$ is a *directed set*, i.e. for every $F_1, F_2 \in \mathcal{F}$ there is an F such that $F \supseteq F_1, F_2$ (obviously, $F = F_1 \cup F_2$ will do in our case, but in general the set $\mathcal{F}$ may be ordered by a relation that has nothing to do with set theory).

For directed sets, there is a suitable notion of a limit. Let $(a_F)_{F\in\mathcal{F}}$ be a family of numbers or vectors in a Banach space and let a be an element in the same space. We will write

$$a = \lim_{F\in\mathcal{F}} a_F$$

iff for every $\epsilon > 0$ there is an element $F_0 \in \mathcal{F}$ such that for every $F \in \mathcal{F}$, $F \succeq F_0$ implies

$$|a_F - a| < \epsilon.$$

(In the case of Banach spaces, $|\cdot|$ would mean the norm.) Note that the limit defined in this way is unique.

In the case of a_F being real numbers, or elements of a partially ordered Banach space, there is a notion of a monotonic sequence. For instance, $(a_F)_{F\in\mathcal{F}}$ is increasing if $F_1 \succeq F_2$ implies $a_{F_1} \geq a_{F_2}$. For real numbers, a bounded monotonic sequence always converges. Many partially ordered vector spaces also have this property of order-completeness. For instance, ℓ^∞ with sequences ordered coordinate-wise is order-complete. It will be convenient to say that

$$\lim_{F\in\mathcal{F}} a_F = \infty$$

when for every a (a number or vector) there is an $F_0 \in \mathcal{F}$ such that for every $F \succeq F_0$ we have $a_F \geq a$. It is true for real numbers that every increasing sequence has a finite or infinite limit. In the case of partially ordered, order-complete vector spaces there is an issue of whether for any increasing sequence $(a_F)_{F\in\mathcal{F}}$ the limit is equal to the supremum (lowest upper bound), i.e.

$$\lim_{F\in\mathcal{F}} a_F = \sup_{F\in\mathcal{F}} a_F.$$

We definitely need to assume that the order in the vector space is a continuous function. If we also assume that supremum exists for every bounded sequence, we essentially assume that our partially ordered vector is a *Banach lattice.*

When the sequence is either real-valued or assumes values in an order-complete partially ordered vector space, the notion of upper and lower limit can be developed. For instance,

$$\begin{aligned}
\limsup_{F\in\mathcal{F}} a_F &\stackrel{\text{def}}{=} \lim_{F_0\in\mathcal{F}} \sup_{F\in\mathcal{F}, F\succeq F_0} a_F = \inf_{F_0\in\mathcal{F}} \sup_{F\in\mathcal{F}, F\succeq F_0} a_F,\\
\liminf_{F\in\mathcal{F}} a_F &\stackrel{\text{def}}{=} \lim_{F_0\in\mathcal{F}} \inf_{F\in\mathcal{F}, F\succeq F_0} a_F = \sup_{F_0\in\mathcal{F}} \inf_{F\in\mathcal{F}, F\succeq F_0} a_F.
\end{aligned}$$

We note that every sequence from above has an upper limit.

Theorem 3.6. *Let $(\mathcal{F}, \succeq)$ be a directed set and let $u_F : [p, q) \to \overline{\mathbb{R}}$, $F \in \mathcal{F}$ be a family of functions, such that the pointwise limit*

$$u = \lim_{F\in\mathcal{F}} u_F$$

exists. Let $0 < \delta < q - p$. For all $t \in [p, q - \delta]$ such that $u(t) \in \mathbb{R}$, we have:

$$D^\dagger_\delta u(t) \leq \liminf_{F\in\mathcal{F}} D^\dagger_\delta u_F(t).$$

Proof. Let us consider a fixed t' such that $t < t' < t + \delta$. We have

$$\frac{u(t') - u(t)}{t' - t} = \lim_{F\in\mathcal{F}} \frac{u_F(t') - u_F(t)}{t' - t}.$$

Also, for any fixed F

$$\frac{u_F(t') - u_F(t)}{t' - t} \leq D^\dagger_\delta u_F(t).$$

Thus,

$$\liminf_{F\in\mathcal{F}} \frac{u_F(t') - u_F(t)}{t' - t} \leq \liminf_{F\in\mathcal{F}} D^\dagger_\delta u_F(t).$$

In view of convergence, the left-hand side is equal to

$$\lim_{F\in\mathcal{F}} \frac{u_F(t') - u_F(t)}{t'-t}$$

and thus we have:

$$\frac{u(t') - u(t)}{t'-t} \le \liminf_{F\in\mathcal{F}} D_\delta^\dagger u_F(t).$$

Hence, also

$$D_\delta^\dagger u(t) = \sup_{t'\in(t,t+\delta)} \frac{u(t') - u(t)}{t'-t} \le \liminf_{F\in\mathcal{F}} D_\delta^\dagger u_F(t).$$

This completes the proof. □

Definition 3.2. *For any family of numbers $(a_l)_{l\in\mathcal{L}}$ and in all cases (finite, countable and uncountable) we define:*

$$\sum_{l\in\mathcal{L}} a_l \stackrel{\text{def}}{=} \lim_{F\in\mathcal{F}(\mathcal{L})} \sum_{l\in F} a_l\,.$$

We say that $(a_l)_{l\in\mathcal{L}}$ is summable iff *the limit exists.*

The reader may regard this expression as definition in the case of uncountable sets $\mathcal{L}$, and for countable sets $\mathcal{L}$ this definition of summability is consistent with the ordinary definition of *absolute convergence* of a series.

Moreover, one can show that even if $\mathcal{L}$ is uncountable then for this generalized sum to converge it is necessary that the set $\{l \in \mathcal{L} \,:\, a_l \neq 0\}$ be countable. Indeed, let $S = \sum_{l\in\mathcal{L}} a_l$. Let $\epsilon_n \searrow 0$. We pick a sequence of finite sets F_n, $n = 1, 2, \ldots$, such that for every n and for every $F \subseteq \mathcal{L}$ such that $F \supseteq F_n$ we have

$$\left|\sum_{l\in F} a_l - S\right| < \epsilon_n.$$

In particular $\lim_{n\to\infty}\sum_{l\in F_n} = S$. Let $\mathcal{L}_0 = \bigcup_{n=1}^\infty F_n$. This set is countable as a countable union of finite sets. We claim that $a_l = 0$ for $l \in \mathcal{L}\backslash\mathcal{L}_0$. Let us prove this by contradiction. Let us suppose that for some $l_0 \in \mathcal{L}\backslash L_0$. Let $G_n = F_n \cup \{l_0\}$. Clearly, $\sum_{l\in G_n} a_l = a_{l_0} + \sum_{l\in F_n} a_l$, and thus $\lim_{n\to\infty}\sum_{l\in G_n} a_l = a_{l_0} + \lim_{n\to\infty}\sum_{l\in F_n} a_l = a_{l_0} + S \neq S$. For sufficiently large n, this contradicts the definition of F_n.

Also, in case of non-negative a_l, we have:

$$\sum_{l\in\mathcal{L}} a_l = \lim_{F\in\mathcal{F}(\mathcal{L})} \sum_{l\in F} a_l = \sup_{F\in\mathcal{F}(\mathcal{L})} \sum_{l\in F} a_l\,.$$

We will need the notion of the *upper sum* of a generalized series of numbers.

Definition 3.3. *Let $(a_l)_{l\in\mathcal{L}}$ be a family of real numbers or elements of a partially ordered, order-complete vector space. Then the* upper sum *of our sequence is*

$$\lim_{F_0\in\mathcal{F}(\mathcal{L})} \sup_{F\supseteq F_0} \sum_{l\in F} a_l = \inf_{F_0\in\mathcal{F}(\mathcal{L})} \sup_{F\supseteq F_0} \sum_{l\in F} a_l.$$

We will denote the upper sum by

$$\sum_{l\in\mathcal{L}}{}^{\dagger}\, a_l.$$

Intuitively speaking, the upper sum is close to the maximal partial sums over large finite subsets. If $u_l : [p,q] \to \overline{\mathbb{R}}$ is a family of functions then the sum $\sum^{\dagger}_{l\in\mathcal{L}} u_l$ is understood pointwise, i.e.

$$\left(\sum_{l\in\mathcal{L}}^{\dagger} u_l\right)(t) \overset{\text{def}}{=} \sum_{l\in\mathcal{L}}^{\dagger} u_l(t).$$

Theorem 3.7. *Let $\mathcal{F}$ be a directed set with the order relation $\succeq$. Let $u_F : [p,q) \to \overline{\mathbb{R}}$, $F \in \mathcal{F}$, be any family of functions (finite, countable or uncountable) and let*

$$u = \limsup_{F\in\mathcal{F}} u_F.$$

Let $0 < \delta < q - p$. For all $t \in [p, q-\delta]$ such that $u(t) \in \mathbb{R}$, we have:

$$D^{\dagger}_{\delta} u(t) \le \limsup_{F\in\mathcal{F}} D^{\dagger}_{\delta} u_F(t).$$

Proof. for every $F_0 \in \mathcal{F}$ we define pointwise:

$$v_{F_0} = \sup_{F\in\mathcal{F}, F\succeq F_0} u_F.$$

By Theorem 3.3, for every $F_0 \in \mathcal{F}$ we have:

$$D^{\dagger}_{\delta} v_{F_0}(t) \le \sup_{F\in\mathcal{F}, F\succeq F_0} D^{\dagger}_{\delta} v_F(t).$$

We notice that $\lim_{F_0\in\mathcal{F}} v_{F_0}$ exists as a limit of a decreasing sequence, and equals u. Therefore, by Theorem 3.6,

$$\begin{aligned}
D^{\dagger}_{\delta} u(t) &\le \liminf_{F_0\in\mathcal{F}} D^{\dagger}_{\delta} v_{F_0} \\
&\le \liminf_{F_0\in\mathcal{F}} \sup_{F\in\mathcal{F}, F\succeq F_0} D^{\dagger}_{\delta} v_F(t) \\
&= \lim_{F_0\in\mathcal{F}} \sup_{F\in\mathcal{F}, F\succeq F_0} D^{\dagger}_{\delta} v_F(t) \\
&= \limsup_{F\in\mathcal{F}} D^{\dagger}_{\delta} v_F(t).
\end{aligned}$$

□

With the notion of upper sum, we are finally able to formulate a subadditivity theorem for the upper derivative.

Theorem 3.8. *Let $u_l : [p,q) \to \mathbb{R}$, $l \in \mathcal{L}$, be any family of functions (finite, countable or uncountable). Let $0 < \delta < q - p$. For any constants $\theta_l \ge 0$, $l \in \mathcal{L}$, and for any $t \in [p, q-\delta]$ such that the sum $\sum^{\dagger}_{l\in\mathcal{L}} \theta_l u_l(t)$ is finite:*

$$D^{\dagger}_{\delta}\left(\sum_{l\in\mathcal{L}}^{\dagger} \theta_l u_l\right)(t) \le \sum_{l\in\mathcal{L}}^{\dagger} \theta_l D^{\dagger}_{\delta} u_l(t).$$

This inequality should be understood in the following sense: if the right-hand side is finite, then the left-hand side is finite and the inequality holds.

Proof. We note that *by definition*, for every subset $\mathcal{L}' \subseteq \mathcal{L}$ the sum of functions $\sum^{\dagger}_{l\in\mathcal{L}'} \theta_l u_l$ is defined by pointwise summation:

$$\left(\sum_{l\in\mathcal{L}'}^{\dagger} \theta_l u_l\right)(t) = \sum_{l\in\mathcal{L}'}^{\dagger} \theta_l u_l(t)\,.$$

For every $F \in \mathcal{F}$, let us define $v_F : [p,q) \to \mathbb{R}$ via:

$$v_F = \sum_{l \in F} \theta_l u_l \,.$$

In view of our definitions,

$$v = \limsup_{F \in \mathcal{F}(\mathcal{L})} v_F = {\sum_{l \in \mathcal{L}}}^{\dagger} \theta_l u_l \,.$$

Also, in view of *finite* subadditivity, we have:

$$D^{\dagger}_{\delta} v_F(t) \le \sum_{l \in F} \theta_l D^{\dagger}_{\delta} u_l(t) \,.$$

By Theorem 3.7,

$$D^{\dagger}_{\delta} u(t) \le \limsup_{F \in \mathcal{F}(\mathcal{L})} D^{\dagger}_{\delta} v_F(t) \le \limsup_{F \in \mathcal{F}(\mathcal{L})} \sum_{l \in F} \theta_l D^{\dagger}_{\delta}(t) = {\sum_{l \in \mathcal{L}}}^{\dagger} \theta_l D^{\dagger}_{\delta}(t).$$

□

In the next lemma it will be useful to have the notion of *left-upper limit* of a function $u : (a-\epsilon, a) \to \overline{\mathbb{R}}$, defined as follows:

$$\limsup_{t \to a(-)} u(t) \stackrel{\text{def}}{=} \lim_{\delta \to 0} \sup_{t \in (a-\delta, a)} u(t).$$

A function $u : (a-\epsilon, a] \to \mathbb{R}$ will be called *left-upper-semicontinuous* at a iff $u(a) \le \limsup_{t \to a(-)} u(t)$. If a function is left-upper-semicontinuous at every point of its domain, it will be simply called left-upper-semicontinuous.

Lemma 3.9. *Let $u : [p,q) \to \mathbb{R}$ be left-upper-semicontinuous on (p,q).*
(a) If there is some constant $K \in \mathbb{R}$ such that for all $t \in [p,q)$:

$$D^{\dagger} u(t) \le K$$

then for every $t_1, t_2 \in [p,q)$ we have

$$u(t_2) - u(t_1) \le K(t_2 - t_1);$$

(b) Let in addition $u \ge 0$, u is continuous. If there is a $c \in \mathbb{R}$ such that for each $t \in [p,q)$

$$D^{\dagger} u(t) \le c u(t)$$

then for every $t_1, t_2 \in [p,q)$, $t_1 \le t_2$, we have:

$$u(t_2) \le u(t_1) \exp(c(t_2 - t_1)).$$

Proof. (a) We note that it is sufficient to prove (a) in the case when $t_1 = p$ because if $t_1 > p$, we may simply replace p with t_1. We will prove that for every $K_1 > K$ and every $t_2 \in [p,q)$

$$u(t_2) - u(p) \le K_1(t_2 - p).$$

Since $K_1 > K$ is otherwise arbitrary in our argument, (a) will follow. Let us define $t_3 \in [p,q]$ in the following way:

$$t_3 = \sup \left\{ t_2 \in (p,q) : \ \forall t \in (p, t_2) \ \frac{u(t) - u(p)}{t - p} \le K_1 \right\}.$$

As for some $\delta > 0$ we have $D^{\dagger}_{\delta} u(p) < K_1$, the set in this definition is non-empty. Thus $t_3 > p$. Since u is left-upper-semicontinuous, we have

$$\frac{u(t_3) - u(p)}{t_3 - p} \leq \limsup_{t \to t_3(-)} \frac{u(t) - u(p)}{t - p} \leq K_1 \,.$$

We claim that $t_3 = q$ which implies (a). Let us prove this statement by contradiction. Thus, let us assume that $t_3 < q$. There exists $\delta > 0$ such that $D^{\dagger}_{\delta} u(t_3) \leq K_1$. From the definition of t_3, there exists $t_4 \in (t_3, t_3 + \delta)$ such that

$$\frac{u(t_4) - u(p)}{t_4 - p} > K_1.$$

On the other hand,

$$\begin{aligned}
\frac{u(t_4) - u(p)}{t_4 - p} &= \frac{u(t_4) - u(t_3)}{t_4 - t_3} \cdot \frac{t_4 - t_3}{t_4 - p} + \frac{u(t_3) - u(p)}{t_3 - p} \cdot \frac{t_3 - p}{t_4 - p} \\
&\leq K_1 \cdot \frac{t_4 - t_3}{t_4 - p} + K_1 \cdot \frac{t_3 - p}{t_4 - t} = K_1.
\end{aligned}$$

This is a contradiction. Thus $t_3 = q$, and the proof of part (a) is complete.
(b) Consider the function $v(t) = \exp(-ct)u(t)$. By the subadditive product rule, we obtain:

$$\begin{aligned}
D^{\dagger} v(t) &\leq -c\exp(-ct)u(t) + \exp(-ct)D^{\dagger}u(t) \\
&\leq -c\exp(-ct)u(t) + \exp(-ct)cu(t) = 0.
\end{aligned}$$

Thus v is decreasing from (a). Hence, for every $t_1, t_2 \in [p, q)$, $t_1 \leq t_2$, we have $u(t_2) \leq u(t_1)\exp(c(t_2 - t_1))$.

□

Lemma 3.10. *Let $\mathbf{y} : [p, q) \to Y$ be a path in a Banach space Y. Let $(\varphi_l)_{l \in \mathcal{L}}$ be a bounded set in Y^*. If $\mathbf{y}(t)$ is continuous at some $t \in [p, q)$ and $\mathbf{y}'(t)$ exists then the function*

$$u = \sup_{l \in \mathcal{L}} (\varphi_l \circ \mathbf{y})$$

possesses the right-upper derivative $D^{\dagger}u(t)$ and

$$D^{\dagger}u(t) \leq \sup_{l \in \mathcal{L}} \varphi_l(\mathbf{y}'(t)).$$

Proof. Let $C = \sup_l \|\varphi_l\|$. By definition of differentiability of $\mathbf{y}(t)$, for every $\epsilon > 0$ there is $\delta > 0$ and a function $\mathbf{z} : (0, \delta) \to Y$ such that if $t' \in (t, t + \delta)$ then

$$\mathbf{y}(t') = \mathbf{y}(t) + \mathbf{y}'(t)(t' - t) + (t' - t)\mathbf{z}(t')$$

where $\|\mathbf{z}(t')\| < \epsilon$ for each $t' \in (t, t + \delta)$. By the linearity of φ_l, for every $l \in \mathcal{L}$ we have:

$$\frac{\varphi_l(\mathbf{y}(t')) - \varphi_l(\mathbf{y}(t))}{t' - t} - \varphi_l(\mathbf{y}'(t)) = \varphi_l(\mathbf{z}(t')).$$

Therefore

$$\left| \frac{\varphi_l(\mathbf{y}(t')) - \varphi_l(\mathbf{y}(t))}{t' - t} - \varphi_l(\mathbf{y}'(t)) \right| = |\varphi_l(\mathbf{z}(t'))| \leq \|\varphi_l\| \|\mathbf{z}(t')\| \leq C\epsilon.$$

In particular,

$$\frac{\varphi_l(\mathbf{y}(t')) - \varphi_l(\mathbf{y}(t))}{t' - t} \leq \varphi_l(\mathbf{y}'(t)) + C\epsilon.$$

Taking the supremum over $l \in \mathcal{L}$ we obtain:

$$\sup_{l \in \mathcal{L}} \frac{\varphi_l(\mathbf{y}(t')) - \varphi_l(\mathbf{y}(t))}{t' - t} \leq \sup_{l \in \mathcal{L}} \varphi_l(\mathbf{y}'(t)) + C\epsilon. \tag{3.1}$$

Also, for any two families of numbers $(a_l)_{l\in\mathcal{L}}$ and $(b_l)_{l\in\mathcal{L}}$ we have:

$$\sup_{l\in\mathcal{L}} a_l - \sup_{l\in\mathcal{L}} b_l \leq \sup_{l\in\mathcal{L}}(a_l - b_l)\,.$$

Thus,

$$\begin{aligned}
\frac{u(t')-u(t)}{t'-t} &= \frac{\sup_{l\in\mathcal{L}}\varphi_l(\mathbf{y}(t')) - \sup_{l\in\mathcal{L}}\varphi_l(\mathbf{y}(t))}{t'-t} \\
&\leq \frac{\sup_{l\in\mathcal{L}}\left[\varphi_l(\mathbf{y}(t')) - \varphi_l(\mathbf{y}(t))\right]}{t'-t} \\
&= \sup_{l\in\mathcal{L}} \frac{\varphi_l(\mathbf{y}(t')) - \varphi_l(\mathbf{y}(t))}{t'-t} \\
&\leq \sup_{l\in\mathcal{L}} \varphi_l(\mathbf{y}'(t)) + C\epsilon.
\end{aligned}$$

(In the last step we used inequality (3.1).)

Hence, from the definition of $D^\dagger u(t)$ and $D^\dagger_\delta u(t)$ we obtain:

$$D^\dagger u(t) \leq D^\dagger_\delta u(t) = \sup_{t'\in(t,t+\delta)} \frac{u(t')-u(t)}{t'-t} \leq \sup_{l\in\mathcal{L}}\varphi_l(\mathbf{y}'(t)) + C\epsilon.$$

As $\epsilon > 0$ was otherwise arbitrary, we also obtain

$$D^\dagger u(t) \leq \sup_{l\in\mathcal{L}}\varphi_l(\mathbf{y}'(t))$$

which is the conclusion of the lemma. □

3.3. **The proof of exponential convergence.**

Lemma 3.11. *For every solution $\mathbf{y}(t)$ of (2.1) defined on an interval $[p,q]$ and every $t \in [p,q]$ we have:*

$$D^\dagger(V\circ\mathbf{y})(t) \leq -\alpha(\mathbf{y}(t))(V\circ\mathbf{y})(t).$$

Proof. We note that

$$V(\mathbf{y}) = \max_{i,j:y_i\geq s\geq y_j}(y_i - y_j).$$

When $V(\mathbf{y}) > 0$, i.e. when $\mathbf{y}$ is a non-equilibrium, we define the set $\mathcal{L} = \mathcal{L}(\mathbf{y})$ to be the set of these pairs of indices (i,j) for which the above maximum is attained, or, equivalently, $y_i = M(\mathbf{y})$ and $y_j = m(\mathbf{y})$. Just as for the earlier defined function $\mathcal{I}(\mathbf{y})$ in Eq. (2.2), the set-valued function $\mathcal{L}(\mathbf{y})$ is upper semicontinuous. Hence, there is $\delta > 0$ such that if $\mathbf{z}\in\mathbb{R}^n$ and $\|\mathbf{z}-\mathbf{y}\|_\infty < \delta$ then $\mathcal{L}(\mathbf{z}) \subseteq \mathcal{L}(\mathbf{y})$. Let us pick $\epsilon > 0$ such that for all $t' \in [p,q]\cap(t-\epsilon,t+\epsilon)$ we have $\mathbf{y}(t') \in B_\delta(\mathbf{y}(t))$. Thus, for all $t' \in [p,q]\cap(t-\epsilon,t+\epsilon)$ we have:

$$V(\mathbf{y}(t')) = \max_{(i,j)\in\mathcal{L}(\mathbf{y}(t))}(y_i - y_j). \tag{3.2}$$

We will apply Lemma 3.10 in order to show that

$$D^\dagger(V\circ\mathbf{y})(t)) = \max_{(i,j)\in\mathcal{L}(\mathbf{y}(t))}(\dot{y}_i - \dot{y}_j).$$

For any $(i,j)\in\mathcal{L}$, let $\varphi_{(i,j)}$ be the linear functional on $\mathbb{R}^n$ defined by the formula:

$$\varphi_{(i,j)}(\mathbf{y}) = y_i - y_j.$$

Clearly, $\|\varphi_{(i,j)}\| = 2$ in the norm of $(\mathbb{R}^n)^*$. Equation (3.2) can be expressed in a form suitable for an application of Lemma 3.10:

$$V(\mathbf{y}(t')) = \max_{(i,j)\in\mathcal{L}(\mathbf{y(t)})}\varphi_{(i,j)}(\mathbf{y}).$$

Thus, Lemma 3.10 implies that

$$D^{\dagger}(V \circ \mathbf{y})(t)) \leq \max_{(i,j)\in\mathcal{L}(\mathbf{y}(t))} \varphi_{(i,j)}(\mathbf{y}'(t)) = \max_{(i,j)\in\mathcal{L}(\mathbf{y}(t))} (\dot{y}_i - \dot{y}_j)$$

With the help of Lemma 3.1 we conclude that

$$\max_{(i,j)\in\mathcal{L}(\mathbf{y}(t))} (\dot{y}_i - \dot{y}_j) \leq -\alpha(\mathbf{y}(t))V(\mathbf{y}(t)),$$

thus

$$D^{\dagger}(V \circ \mathbf{y})(t)) \leq -\alpha(\mathbf{y}(t))V(\mathbf{y}(t)).$$

□

There is a more geometric interpretation of the function V and it is provided in the following simple lemma.

Lemma 3.12. *Let N be the the ray of equilibria of system 2.1, i.e.*

$$N = \{(c, c, \ldots, c) : c \in \mathbb{R}, c > 0\}$$

Let $\operatorname{dist}(\mathbf{y}, N)$ *be the distance of* $\mathbf{y}$ *from* N*, measured with respect to the norm* $\|\cdot\|_\infty$*, i.e.*

$$\operatorname{dist}(\mathbf{y}, N) = \inf_{\mathbf{z}\in N} \|\mathbf{y} - \mathbf{z}\|_\infty.$$

Let U be the set of positive vectors in $\mathbb{R}^n$:

$$U = \{\mathbf{y} \in \mathbb{R}^n : \forall i \in \{1, 2, \ldots, n\}\ y_i > 0\}.$$

Then for every $\mathbf{y} \in U$ we have the following alternative definition of $V(\mathbf{y})$:

$$V(\mathbf{y}) = 2\operatorname{dist}(\mathbf{y}, N).$$

Proof. Indeed, it is easy to see that the minimum distance in the $\|\cdot\|_\infty$ norm between $\mathbf{y}$ and N is achieved for $\mathbf{z} \in N$ for which $z_i = \frac{1}{2}(M(\mathbf{y}) + m(\mathbf{y}))$ for $i = 1, 2, \ldots, n$. It is easy to see that

$$\|\mathbf{y} - \mathbf{z}\|_\infty = \frac{1}{2}(M(\mathbf{y}) - m(\mathbf{y})) = \frac{1}{2}V(\mathbf{y}).$$

□

Theorem 3.13. *(a) For every initial condition $\mathbf{y}_0 \in U$ there is a unique solution $\mathbf{y}(t, \mathbf{y}_0)$ of (2.1) with the initial condition $\mathbf{y}_0$ (i.e. $\mathbf{y}(0, \mathbf{y}_0) = \mathbf{y}_0$) defined on the interval $[0, \infty)$, and for all $t \geq 0$, $\mathbf{y}(t, \mathbf{y}_0) \in U$. Thus, there exists a* semi-flow *$\varphi^t : U \to U$, $t \geq 0$ defined by the formula*

$$\varphi^t(\mathbf{y}_0) = \mathbf{y}(t, \mathbf{y_0}).$$

(b) Let $\mathbf{y}_0 \in U$ be an initial condition and let α_0 be the following constant:

$$\alpha_0 = \frac{2\min_i \lambda_i}{M(\mathbf{y}_0)^3}.$$

There is a unique equilibrium $\bar{\mathbf{y}}$ such that if $\mathbf{y}(t, \mathbf{y}_0)$ is the solution of (2.1) with the initial condition $\mathbf{y}_0$ defined on the interval $[0, \infty)$, then for all $t \geq 0$:

$$\|\mathbf{y}(t, \mathbf{y}_0) - \bar{\mathbf{y}}\|_\infty \leq (M(\mathbf{y}_0) - m(\mathbf{y}_0))\exp(-\alpha_0 t) = 2\operatorname{dist}(\mathbf{y}_0, N)\exp(-\alpha_0 t). \tag{3.3}$$

(c) The mapping $r : \mathbf{y}_0 \mapsto \bar{\mathbf{y}}$ is a continuous retraction of U onto the ray of equilibria

$$N = \{(c, c, \ldots, c) : c \in \mathbb{R}, c > 0\},$$

i.e. it is a continuous map $r : U \to N$ such that $r|N = id_N$. Moreover, for every $\mathbf{y}_0 \in U$:

$$\|\mathbf{y}_0 - r(\mathbf{y}_0)\|_\infty \leq 2\operatorname{dist}(\mathbf{y}_0, N).$$

(d) Let $\mathbf{F} : U \to \mathbb{R}^n$ be the vector field of the system (2.1), i.e. $\mathbf{F}(\mathbf{y}) = (F_1(\mathbf{y}), F_2(\mathbf{y}), \ldots, F_n(\mathbf{y}))$ and the coordinate mappings $F_i : U \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are given by the formulas:

$$F_i(\mathbf{y}) = \lambda_i \left(-\frac{1}{s^2} + \frac{1}{y_i^2} \right).$$

Let $\bar{\mathbf{y}} \in N$ be arbitrary and let $\sigma = \sigma(D\mathbf{F}(\bar{\mathbf{y}}))$ be the spectrum of $D\mathbf{F}(\bar{\mathbf{y}})$ (the Frechét derivative). Then $\sigma = \{0\} \cup \sigma_0$, where

$$\sigma_0 \subset \{z \in \mathbb{C} : \Re(z) \leq -\alpha_0\}.$$

The zero eigenvalue is simple and it corresponds to the eigenvector $\mathbf{v}_0 = (1, 1, \ldots, 1)$, which is tangent to the ray of equilibria N, while σ_0 is the spectrum of $D\mathbf{F}(\bar{\mathbf{y}})$ restricted to some eigenspace V of codimension 1 transversal to the line of equilibria. In particular, such an eigenspace exists.

Proof. (a) The forward completeness of the solution has been proved in Theorem 2.1 and thus a semi-flow for every initial condition $\mathbf{y}_0 \in U$ exists..
(b) For any two numbers $\bar{m} < \bar{M}$ we define

$$P(\bar{m}, \bar{M}) = \{\mathbf{y} \in \ell^\infty \, : \, \bar{m} \leq m(\mathbf{y}) \text{ and } M(\mathbf{y}) \leq \bar{M}\}.$$

In view of the fact that $M(\mathbf{y}(t))$ and $m(\mathbf{y}(t))$ are monotonic, for every $t' \in [t, \infty)$

$$\mathbf{y}(t') \in P(m(\mathbf{y}(t)), M(\mathbf{y}(t))).$$

Also, $\lim_{t\to\infty} M(\mathbf{y}(t)) = \lim_{t\to\infty} m(\mathbf{y}(t))$ from Section 2. Thus,

$$\{\bar{\mathbf{y}}\} = \bigcap_{t\geq 0} P(m(\mathbf{y}(t)), M(\mathbf{y}(t))).$$

We also observe that the ℓ^∞-diameter of $P(\bar{m}, \bar{M})$ is $\bar{M} - \bar{m}$, in particular,

$$\|\mathbf{y}(t) - \bar{\mathbf{y}}\|_\infty \leq M(\mathbf{y}(t)) - m(\mathbf{y}(t)) = V(\mathbf{y}(t)).$$

Since $M(\mathbf{y}(t)) < M(\mathbf{y}_0)$, we have

$$-\alpha(\mathbf{y}(t)) < -\alpha_0$$

for all $t > 0$. From Lemma 3.11,

$$D^\dagger(V \circ \mathbf{y})(t) \leq -\alpha_0 \cdot (V \circ \mathbf{y})(t).$$

Besides, $V(\mathbf{y}(t)) = M(\mathbf{y}(t)) - m(\mathbf{y}(t))$ is continuous because $M(\mathbf{y}(t))$ and $m(\mathbf{y}(t))$ are obviously continuous. From Lemma 3.9, we have

$$V(\mathbf{y}(t)) \leq V(\mathbf{y}_0) \exp(-\alpha_0 t)\,.$$

Therefore, the proof of part (b) is complete.
(c) The only part needing a proof is that of the continuity of r. But this follows easily from the inequality (3.3) of part (*b*). This inequality means that $r(\mathbf{y}_0) = \lim_{t\to\infty} \varphi^t(\mathbf{y}_0)$ and the limit is locally uniform on U. Thus, r is continuous as a locally uniform limit of continuous functions.
(d) We recall that

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

is well-defined for every bounded linear operator A. We have the following well-known equation, connecting the linearization of the flow with the linearization of the vector field at an equilibrium (see [2], or [1] which also covers the infinite-dimensional case):

$$D\varphi^t(\bar{\mathbf{y}}) = \exp(tD\mathbf{F}(\bar{\mathbf{y}})). \tag{3.4}$$

Since our model is a homogeneous system, it is true that any vector tangent to N (which is also in N in this case) is an eigenvector of $D\mathbf{F}(\bar{\mathbf{y}})$ with eigenvalue 0 (see [7]), and therefore an eigenvector of

$D\varphi^t(\bar{\mathbf{y}})$ with eigenvalue 1 from (3.4). As the vector $\mathbf{v}_0 = (1, 1, \ldots, 1) \in N$ is an eigenvector of both $D\varphi^t(\bar{\mathbf{y}})$ (with eigenvalue 1) and of $D\mathbf{F}(\bar{\mathbf{y}})$ (with eigenvalue 0), we may consider the corresponding induced linear operators on the quotient space $W = \mathbb{R}^n/N = \mathbb{R}^n/\operatorname{span}\{\mathbf{v}_0\}$. We will use the same symbol for the induced operator as for the original operator. Let σ_0 be the spectrum of the operator induced on W by $D\mathbf{F}(\bar{\mathbf{y}})$. We claim that $\sigma_0 \subseteq \{z : \Re z \leq -\alpha_0\}$, which implies all other claims, in particular, simplicity of 0 and the spectral decomposition $\sigma = \{0\} \cup \sigma_0$. We will prove our claim by contradiction. If the claim is false for some $\bar{\mathbf{y}} \in N$ then the spectral radius of $D\varphi^t(\bar{\mathbf{y}})$, which is equal to $\exp\left(t \sup_{z\in\sigma_0} \Re z\right)$, is greater than $\exp(-t\alpha_0)$. Let us pick $\alpha_1 < \alpha_0$ such that the spectral radius of $D\varphi^t(\bar{\mathbf{y}})$ is also greater than $\exp(-t\alpha_1)$ for $t = 1$. We know that the spectral radius of a bounded linear operator A can be also expressed as $\limsup_{k\to\infty} \|A^k\|^{1/k}$ (e.g., see [8]). We will assume that all norms are calculated with respect to the norm induced by $\|\cdot\|_\infty$, i.e. for any vector $\mathbf{w} \in W$:

$$\|\mathbf{w}\| = \inf_{\mathbf{v}\in\mathbf{w}} \|\mathbf{v}\|_\infty.$$

The reader should remember that $\mathbf{w}$ is an equivalence class of vectors, where two vectors $\mathbf{v}_1, \mathbf{v}_2$ are equivalent if there is an $s \in \mathbb{R}$ such that $\mathbf{v}_1 - \mathbf{v}_2 = s\mathbf{v}_0$. Thus, the notation $\mathbf{v} \in \mathbf{w}$ makes sense. Furthermore,

$$\limsup_{t\to\infty} \|D\varphi^t(\bar{\mathbf{y}})\|^{1/t} \geq \limsup_{n\to\infty} \|D\varphi^n(\bar{\mathbf{y}})\|^{1/n} = \limsup_{n\to\infty} \|\left(D\varphi^1(\bar{\mathbf{y}})\right)^n\|^{1/n} > \exp(-\alpha_1),$$

where the equality is from the fact that

$$\exp(A + B) = \exp(A)\exp(B)$$

whenever the bounded linear operators A and B commute.

Thus, there exists arbitrarily large t such that

$$\|D\varphi^t(\bar{\mathbf{y}})\| > \exp(-\alpha_1 t).$$

For reasons which will be clear shortly, we fix a t so that

$$\exp(-\alpha_1 t) > 2\exp(-\alpha_0 t)$$

which is possible in view of $\alpha_1 < \alpha_0$. More explicitly, we pick $t > \ln 2/(\alpha_0 - \alpha_1)$.

Hence, from the definition of the induced norm of a bounded linear operator there is a non-zero vector $\mathbf{w} \in W$ such that

$$\|D\varphi^t(\bar{\mathbf{y}})\mathbf{w}\| > \exp(-\alpha_1 t)\|\mathbf{w}\|.$$

Let $\mathbf{v} \in \mathbf{w}$ be a vector of minimal norm. We have *for all* $s \in \mathbb{R}$:

$$\|D\varphi^t(\bar{\mathbf{y}})\mathbf{v} + s\mathbf{v}_0\|_\infty > \exp(-\alpha_1 t)\|\mathbf{v}\|_\infty.$$

We also have for every $\epsilon > 0$ (using differentiability of φ^t for a fixed t only):

$$\begin{aligned}\varphi^t(\bar{\mathbf{y}} + \epsilon\mathbf{v}) &= \varphi^t(\bar{\mathbf{y}}) + D\varphi^t(\bar{\mathbf{y}})(\epsilon\mathbf{v}) + o(\epsilon) \\ &= \bar{\mathbf{y}} + \epsilon D\varphi^t(\bar{\mathbf{y}})(\mathbf{v}) + o(\epsilon).\end{aligned}$$

Hence, for every $s \in \mathbb{R}$:

$$\|\varphi^t(\bar{\mathbf{y}} + \epsilon\mathbf{v}) - \bar{\mathbf{y}} + s\mathbf{v}_0\|_\infty \geq \|\epsilon D\varphi^t(\bar{\mathbf{y}})(\mathbf{v}) + s\mathbf{v}_0 + o(\epsilon)\| \geq \epsilon\exp(-\alpha_1 t)\|\mathbf{v}\|_\infty + o(\epsilon).$$

Let us choose $s = s(\epsilon)$ so that $\bar{\mathbf{y}} - s\mathbf{v}_0 = r(\bar{\mathbf{y}} + \epsilon\mathbf{v}_0)$. In fact, $s = -\epsilon$. By the inequality of part (b),

$$\|\varphi^t(\bar{\mathbf{y}} + \epsilon\mathbf{v}) - \bar{\mathbf{y}} + s\mathbf{v}_0\|_\infty \leq 2\operatorname{dist}(\bar{\mathbf{y}} + \epsilon\mathbf{v}, N)\exp(-\alpha_0 t) \leq 2\epsilon\|\mathbf{v}\|_\infty\exp(-\alpha_0 t),$$

where the last inequality comes from the fact that

$$\epsilon\|\mathbf{v}\| = \epsilon\min_{\mathbf{z}\in N}\|\mathbf{v} + \mathbf{z}\| = \min_{\mathbf{z}\in N}\|\epsilon\mathbf{v} + \mathbf{z}\| = \min_{\mathbf{z}\in N}\|\bar{\mathbf{y}} + \epsilon\mathbf{v} - \mathbf{z}\| = \operatorname{dist}(\bar{\mathbf{y}} + \epsilon\mathbf{v}, N).$$

Combining the two inequalities, we obtain

$$\epsilon \exp(-\alpha_1 t)\|\mathbf{v}\|_\infty + o(\epsilon) \le 2\epsilon \|\mathbf{v}\|_\infty \exp(-\alpha_0 t).$$

Dividing by $\epsilon\|\mathbf{v}\|_\infty$ and passing with $\epsilon \to 0$ we obtain:

$$\exp(-\alpha_1 t) \le 2\exp(-\alpha_0 t).$$

This inequality contradicts our choice of t, and thus the proof of part (d) of our theorem is complete. $\square$

Remark 3.14. Another proof of the spectral estimate of part (d), based on the theory of normally hyperbolic invariant manifolds, can be found in [7]. However, the reader may find the direct argument included in the above proof more elementary.

4. A non-shrinking theorem

Even if the attractivity condition $\sqrt{b} = \sum_{j=1}^n \sqrt{\beta_j}$ is not satisfied, but instead just $\sqrt{b} \le \sum_{j=1}^n \sqrt{\beta_j}$, then the economy based on the labor managed oligopoly does not shrink, i.e. the production never goes to 0.

Lemma 4.1. *Let us consider the system (2.1) and the relevant notations. Let us assume $\sqrt{b} \le \sum_{j=1}^n \sqrt{\beta_j}$ so that*

$$\gamma \stackrel{\text{def}}{=} \sum_{j=1}^n \gamma_j \ge 1.$$

Let us define a function $r : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$r(\mathbf{y}) = \sum_{j=1}^n \frac{\gamma_j}{\lambda_j} y_j.$$

Then $\dot{r} \ge 0$ and, at all non-equilibrium points $\dot{r} > 0$.

Proof.

$$\begin{aligned}\dot{r} &= \sum_{j=1}^n \gamma_j \left(-\frac{1}{s^2} + \frac{1}{{y_j}^2}\right) \\ &= -\frac{1}{s^2} + \sum_{j=1}^n \frac{\gamma_j}{y_j^2}\end{aligned}$$

The function

$$\sigma(y) = \frac{1}{(\gamma y)^2}$$

is a strictly convex function, and thus for any constants $\theta_j \ge 0$ such that $\sum_{j=1}^n \theta_j = 1$ we have $\sigma(\sum_{j=1}^n \theta_j y_j) \ge \sum_{j=1}^n \theta_j \sigma(y_j)$. Moreover, if $\theta_j > 0$ for all j then equality holds only if all y_j are equal, i.e. when $\mathbf{y}$ is an equilibrium-point. In particular, if we set $\theta_j = \gamma_j/\gamma$, we obtain.

$$\begin{aligned}\frac{1}{s^2} &= \sigma(\sum_{j=1}^n \theta_j y_j) \le \sum_{j=1}^n \theta_j \sigma(y_j) \\ &= \frac{1}{\gamma}\sum_{j=1}^n \frac{\gamma_j}{(\gamma y_j)^2} = \frac{1}{\gamma^3}\sum_{j=1}^n \frac{\gamma_j}{y_j^2} \le \sum_{j=1}^n \frac{\gamma_j}{y_j^2}\end{aligned}$$

since $\gamma \ge 1$. Moreover, if $\mathbf{y}$ is not an equilibrium point, the inequality is sharp. Our lemma follows. $\square$

Theorem 4.2. *Let us assume $\sqrt{b} \le \sum_{j=1}^{n} \sqrt{\beta_j}$. Then the total output of the industry s does not go to 0 as $t \to \infty$.*

Proof. This is because

$$s(\mathbf{y}) = \sum_{j=1}^{n} \gamma_j y_j = \sum_{j=1}^{n} \lambda_j (\frac{\gamma_j}{\lambda_j} y_j) \ge (\min_j \lambda_j)\, r(\mathbf{y})$$

and $r(\mathbf{y})$ increases along the trajectories of our system. □

5. A generalization to infinitely many firms

In this section we formulate a model analogous to the dynamic model of the previous section, in which $n = \infty$. While the relevance to modeling economic situations may be of moderate interest, this model may be relevant in other applications as a general competition model.

Thus, we will study the system of differential equations

$$\dot{x}_i = k_i \left(-\frac{b}{a_i\, s^2} + \frac{\beta_i}{a_i\, x_i^2} \right) \qquad i = 1, 2, \ldots \tag{5.1}$$

where $k_i > 0$ is a constant specified for each i, and $s = \sum_{i=1}^{\infty} x_i$. As we allow i to go to infinity, we must assume that the sequence x_i is summable, in order for s to be finite. For the moment, we will not consider the conditions for the above system to have a solution, but the reader should realize that for an infinite system of differential equations the existence is not automatic.

The conditions for the sequence $\mathbf{x}$ to be an equilibrium, at least formally, are:

$$-\frac{b}{a_i\, s^2} + \frac{\beta_i}{a_i\, x_i^2} = 0$$

which yields

$$x_i = \frac{\sqrt{\beta_i}}{\sqrt{b}} s$$

for each $i \ge 0$, just as in the case of finite n. We note that in order to ensure consistency with s being the sum of all x_i, we must assume that

$$\sum_{i=1}^{\infty} \frac{\sqrt{\beta_i}}{\sqrt{b}} = 1$$

Based on the clues provided by the finite-dimensional case, we introduce the constants

$$\begin{aligned} \lambda_i &= \frac{\sqrt{b}}{\sqrt{\beta_i}} \frac{k_i b}{a_i}, \\ \gamma_i &= \frac{\sqrt{\beta_i}}{\sqrt{b}} \end{aligned}$$

and we will assume that

$$\sum_{i=1}^{\infty} \gamma_i = 1.$$

Following the case of finite n we introduce new variables y_i related to x_i via the following relations:

$$y_i = \frac{\sqrt{b}}{\sqrt{\beta_i}} x_i = \frac{x_i}{\gamma_i}.$$

We will also consider our model in new coordinates:

$$\begin{aligned} \dot{y}_i &= \lambda_i\left(-\frac{1}{s^2}+\frac{1}{y_i^2}\right), \\ s &= \sum_{i=1}^{\infty}\gamma_i y_i. \end{aligned} \tag{5.2}$$

We also introduce the following two functions:

$$\begin{aligned} M(\mathbf{y}) &= \sup_{i\geq 1} y_i, \\ m(\mathbf{y}) &= \inf_{i\geq 1} y_i. \end{aligned}$$

The use of supremum and infimum is now necessary because of the infinite range of the index i.

We observe that M and m can assume value ∞ without imposing additional assumptions on x_i. In order to be able to carry over our analysis, we will need M and m to be finite. We note that if M is finite then also m is. Thus, we consider only sequences (y_i) for which $M(\mathbf{y})$ is bounded. It is in order now to introduce a Banach space of all (not only positive) sequences $\mathbf{y}$ for which the following quantity:

$$\|\mathbf{y}\|_\infty = \sup_{i\geq 1}|y_i|$$

is finite. We will denote this space by l^∞, which is standard. It will also be convenient to have a notation for a ball of radius r in this space:

$$B_r(\mathbf{y}) = \{\mathbf{z}\in\ell^\infty \,:\, \|\mathbf{z}-\mathbf{y}\|_\infty < r\}.$$

We note that if $\mathbf{y}\in\ell^\infty$ then also s is finite, because the series $\sum_i x_i = \sum_i \gamma_i y_i$ is absolutely convergent. Indeed,

$$\sum_i |x_i| = \sum_i \gamma_i|y_i| \leq \sum_i \gamma_i\|\mathbf{y}\|_\infty = \|\mathbf{y}\|_\infty.$$

We are ready now to discuss the existence and uniqueness of the infinite system of differential equations (5.1). First, we need some rudiments of the theory of differential equations in Banach spaces. Let X be a Banach space with norm $\|\cdot\|$. A differential equation in an set U of a Banach space X is an equation of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$$

where $\mathbf{F} : U\times\mathbb{R}\to X$ is a function. In order to have the *local* existence and uniqueness of solutions, we need $\mathbf{F}$ to be continuous in both variables and to be Lipschitz in variable $\mathbf{x}$. This means that there is a constant L such that

$$\|\mathbf{F}(\mathbf{x}_1, t) - \mathbf{F}(\mathbf{x}_2, t)\| \leq L\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

The above condition must hold for every pair $\mathbf{x}_1$, $\mathbf{x}_2$ of elements of U and each $t\in\mathbb{R}$. A short survey of the standard method of proof is included in Appendix A.

Our model is an autonomous system, that is, $\mathbf{F}$ is a function of $\mathbf{x}$ only. Let us consider such an $\mathbf{F} : U\to X$. If $\mathbf{F}$ is differentiable at every point of U and if U is a convex set (i.e. every two points can be connected with a line segment) then the Mean Value Theorem (which works in Banach spaces) yields:

$$L = \sup_{\mathbf{x}\in U}\|D\mathbf{F}(\mathbf{x})\|.$$

The benefit of having the Banach space formalism for differential equations is that we can write an infinite system of differential equations as a single differential equation. Of course, in our situation the Banach space $X = \ell^\infty$. The function $\mathbf{F}$ is defined by the right-hand sides of the system of differential equations (5.1), i.e. $\mathbf{F}(\mathbf{y}) = \mathbf{z}$, where

$$z_i = \lambda_i \left(-\frac{1}{s^2} + \frac{1}{y_i^2} \right). \tag{5.3}$$

In order to utilize the theory of ordinary differential equations in Banach spaces, we need this function to be well defined as function $\mathbf{F} : U \to X$ and differentiable. The function is well defined when the above sequence $\mathbf{z} = (z_i)_{i=1}^\infty$ belongs to ℓ^∞ whenever $\mathbf{y}$ does. As s does not depend on i, The sequence z_i is bounded only if

$$\inf_i \frac{|y_i|}{\sqrt{\lambda_i}} > 0.$$

We will assume that $\sup_i \lambda_i < \infty$. Further rationale for this condition will be given below. With this condition on λ_i, we may not allow $|y_i|$ to approach 0. In other words, F is only defined on the open subset of ℓ^∞ given by the inequality

$$m(\mathbf{y}) > 0.$$

The candidate for the derivative is an infinite matrix analogous to the Jacobi matrix known from standard multi-variable calculus class:

$$D\mathbf{F}(\mathbf{y}) = \left[\frac{\partial z_i}{\partial y_j} \right].$$

We have the following expression for the partial:

$$\frac{\partial z_i}{\partial y_j} = \begin{cases} \lambda_i \left(\dfrac{2\gamma_i}{s^3} - \dfrac{2}{y_i^3} \right) & \text{if } i = j, \\ \lambda_i \left(\dfrac{2\gamma_j}{s^3} \right) & \text{otherwise.} \end{cases}$$

Differentiability requires that this matrix be a bounded linear operator. For our special case of $X = \ell^\infty$, this is equivalent to the rows of the matrix to be uniformly bounded in the dual space to our Banach space. More precisely, the norm of the matrix $A = [a_{ij}]$ can be calculated according to the formula:

$$\|A\| = \sup_i \sum_j |a_{ij}|.$$

If this norm is finite then the matrix product of a A and a vector $\mathbf{h} \in \ell^\infty$ converges, and $\|A\mathbf{h}\|_\infty \leq \|A\|\|\mathbf{h}\|_\infty$, and thus A defines a bounded linear operator. The dual space to ℓ^∞ is the space ℓ^1 of sequences $\mathbf{w}$ such that $\|\mathbf{w}\|_1 = \sum_{i=1}^\infty |w_i|$ is finite. We remind the reader that this condition is necessary and sufficient for the expression $\langle \mathbf{w}, \mathbf{y} \rangle = \sum_i w_i y_i$ to play the role of the usual scalar product to establish 1:1 correspondence between sequences $\mathbf{w}$ and bounded linear functionals on ℓ^∞. The linear functional corresponding to $\mathbf{w}$ is given by $\langle \mathbf{w}, \cdot \rangle$.

Based on the prior discussion, we can establish conditions for the operator $D\mathbf{F}(\mathbf{x})$ to be bounded. First of all, we notice that we have no assumption so far to control the smallness of the denominators in the partials $\frac{\partial z_i}{\partial y_j}$. Thus, we will require that on U the function m be bounded from below. From now on we will consider U to be the subset of those sequences $\mathbf{y}$ for which $m(\mathbf{y}) > 0$. We also define the following sequence of subsets of U:

$$U_C = \left\{ \mathbf{y} \in \ell^\infty : \frac{1}{C} \leq m(\mathbf{y}) \leq M(\mathbf{y}) \leq C \right\} \tag{5.4}$$

where C is a positive, finite constant. Clearly, if $C' > C$ then $U_{C'} \supset U_C$ and $U = \bigcup_{C>0} U_C$.

The set U_C is a closed and bounded subset of ℓ^∞. Moreover, its interior is the open set:

$$\left\{\mathbf{y} \in \ell^\infty : \frac{1}{C} < m(\mathbf{y}) \le M(\mathbf{y}) < C\right\}.$$

It turns out that now we have enough information to prove the required existence and uniqueness of solutions.

Lemma 5.1. *Let us assume that* $\sup_i \lambda_i < \infty$. *Then on the open, convex subset*

$$U = \{\mathbf{y} \in \ell^\infty : m(\mathbf{y}) > 0\}$$

of ℓ^∞, *the function* $\mathbf{F} : U \to \ell^\infty$ *defined in (5.3) is of class* C^1. *As such, it is locally Lipschitz. Thus, the differential equation*

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y})$$

satisfies the assumptions of the local existence and uniqueness theorem for Banach spaces.

Proof. We notice that $s = \sum_i \gamma_i y_i \ge \sum_i \gamma_i m(\mathbf{y}) = m(\mathbf{y})$. Also $y_i \ge m(\mathbf{y})$ for each i. Thus, we have the following bounds on the partials:

$$\left|\frac{\partial z_i}{\partial y_j}\right| \le \begin{cases} \lambda_i \left(2\frac{\gamma_i}{m(\mathbf{y})^3} + 2\frac{1}{m(\mathbf{y})^3}\right) & \text{if } i = j, \\ \lambda_i \left(2\frac{\gamma_j}{m(\mathbf{y})^3}\right) & \text{otherwise.} \end{cases}$$

Based on these inequalities, we may estimate the norm

$$\begin{aligned} \|D\mathbf{F}(\mathbf{y})\| &\le \sup_i \lambda_i \left(\sum_j 2\frac{\gamma_j}{m(\mathbf{y})^3} + 2\frac{1}{m(\mathbf{y})^3}\right) \\ &= \sup_i 4\lambda_i \frac{1}{m(\mathbf{y})^3} \le 4C^3 \sup_i \lambda_i. \end{aligned}$$

Thus, in order to guarantee the local existence and uniqueness of solutions we will assume that the constants λ_i are uniformly bounded in i, or, in terms of the original constants, that

$$\sup_{1 \le i < \infty} \frac{k_i}{a_i \sqrt{\beta_i}} < \infty.$$

We note that the above argument has shown the boundedness of the formally defined operator $D\mathbf{F}(\mathbf{y})$, but it still needs to be shown that this operator is indeed the derivative of $\mathbf{F}(\mathbf{y})$ at $\mathbf{y}$, i.e. we need to show that

$$\|\mathbf{F}(\mathbf{y} + \mathbf{h}) - \mathbf{F}(\mathbf{y}) - D\mathbf{F}(\mathbf{y})\mathbf{h}\| = o(\|\mathbf{h}\|), \quad \text{as } \mathbf{h} \to \mathbf{0}.$$

Although with some calculations one can easily derive the necessary estimates, it is easier to see that $\mathbf{F}$ is Frechét differentiable from the differentiability of its component functions. First, we note that $\mathbf{F} = \mathbf{\Lambda} \circ \mathbf{F}_1$, where $\Lambda : \ell^\infty \to \ell^\infty$ is a linear diagonal operator given by $\mathbf{\Lambda}(\mathbf{y}) = (\lambda_i y_i)$. This operator is bounded when $\sup_i \lambda_i$ is bounded, which we assume. Thus, differentiability follows if we show that $\mathbf{F}_1$ is differentiable, where

$$\mathbf{F_1}(\mathbf{y}) = \frac{1}{s(\mathbf{y})^2}\mathbf{e} + \mathbf{F}_2(\mathbf{y})$$

Here $\mathbf{e} = (1, 1, 1, \ldots) \in \ell^\infty$ and

$$\mathbf{F}_2(\mathbf{y}) = \left(\frac{1}{y_1^2}, \frac{1}{y_2^2}, \ldots\right).$$

The function $s = s(\mathbf{y})$ is a bounded linear functional on ℓ^∞, and as such is differentiable. We also have $\frac{1}{s(\mathbf{y})^2} = (\xi \circ s)(\mathbf{y})$, where $\xi : \mathbb{R} \to \mathbb{R}$ is given by $\xi(x) = \frac{1}{x^2}$, then by differentiability of compositions, we obtain that $\frac{1}{s(\mathbf{y})^2}$ is differentiable on U. We use the fact that $s \neq 0$ on U. The function $\frac{1}{s(\mathbf{y})^2}\mathbf{e}$ is a product of a differentiable (scalar) function and a (vector) constant, and as such it is differentiable. Finally, we need to show that $\mathbf{F}_2 : \ell^\infty \to \ell^\infty$ is Frechét differentiable. This part of the argument is best done directly, with the help of the Taylor formula of order 1 with remainder in Cauchy form. We are going to show that $D\mathbf{F}_2(\mathbf{y}) = B$ where the linear operator B is given by:

$$B\mathbf{h} = \left(-\frac{2}{y_i^3}h_i\right)$$

or, using the diagonal matrix notation,

$$B = \operatorname{diag}\left(-\frac{2}{y_i^3}\right).$$

Clearly, if $m(\mathbf{y}) > 0$, this is a bounded linear operator on ℓ^∞, as a diagonal operator with bounded entries. For every vector h there exist constants $\theta_j \in [0,1]$, $j = 1, 2, \ldots$ such that:

$$\mathbf{F}_2(\mathbf{y}+\mathbf{h}) - \mathbf{F}_2(\mathbf{y}) - B\mathbf{h} = \left(\frac{6}{(y_i+\theta_i h_i)^4}h_i^2\right)$$

Thus, if $m(\mathbf{y}) > 0$ and $\|\mathbf{h}\| < m(\mathbf{y})$, the above expression is $O(\|\mathbf{h}\|_\infty^2)$, which implies differentiability at $\mathbf{y}$ and $D\mathbf{F}_2(\mathbf{y}) = B$.

In order to prove that $\mathbf{F}$ is C^1, we need to show that $D\mathbf{F}(\mathbf{y})$ is continuous. We can show continuity by estimates specific to $\mathbf{F}$, but it is more prudent to have a more general argument. By general principles of calculus on Banach spaces, the following operations preserve the C^1 property:

(1) A sum of two C^1 functions $f : U \to Y$ and $g : U \to Y$, where X and Y are Banach spaces, and $U \subseteq X$ is an open set; the sum is a C^1 function $h : U \to Y$ defined by $h(x) = f(x)+g(x)$;
(2) A scalar product of a scalar C^1 function $f : U \to \mathbb{R}$ and a Banach space valued C^1 function $g : U \to Y$, where X and Y are arbitrary Banach spaces and $U \subseteq X$ is an open set; the scalar product is a C^1 function $h : U \to Y$ defined by $h(x) = f(x) \cdot g(x)$;
(3) A composition of C^1 functions $f : U \to V$ and $g : V \to Z$, where $U \subseteq X$, $V \subseteq Y$ are open subsets and X, Y, and Z are Banach spaces; the composition is a C^1 function defined by $h(x) = g(f(x)) = (g \circ f)(x)$;
(4) The inverse of a scalar C^1 function $f : U \to \mathbb{R}$, where $U \subseteq X$, X is a Banach space and for every $x \in U$ we have $f(x) \neq 0$; the inverse is a function $h : U \to \mathbb{R}$ defined by $h(x) = \frac{1}{f(x)}$;
(5) The constant function $f : X \to Y$, where X and Y are Banach spaces, is C^1; thus, there is a vector $c \in Y$ such that $f(x) = c$ for all $x \in U$.

As the function $\mathbf{F}$ was expressed in terms of functions s, ξ, a constant function with value $\mathbf{e}$, $\mathbf{F}_2$ and the above considered operations, we only need to show that $\mathbf{F_2}$ is C^1, as other functions are obviously C^1. We have already proved that $\mathbf{F}_2$ is differentiable, and thus it suffices to prove the continuity of $D\mathbf{F}_2$. We have

$$\begin{aligned}
D\mathbf{F}_2(\mathbf{y}+\mathbf{h}) - D\mathbf{F}_2(\mathbf{y}) &= \operatorname{diag}\left(-\frac{2}{(y_i+h_i)^3}\right) - \operatorname{diag}\left(-\frac{2}{y_i^3}\right) \\
&= \operatorname{diag}\left(-\frac{2}{(y_i+h_i)^3} + \frac{2}{y_i^3}\right) \\
&= \operatorname{diag}\left(\frac{6}{(y_i+\theta_i h_i)^4}h_i\right),
\end{aligned}$$

where $\theta_i \in [0,1]$ are constants whose existence is ascertained by the Mean Value Theorem. The norm of a diagonal operator on ℓ^∞ is the supremum of the norms of diagonal entries, and thus for

$\|\mathbf{h}\|_\infty < m(\mathbf{y})$ we have:

$$\|D\mathbf{F}_2(\mathbf{y}+\mathbf{h}) - D\mathbf{F}_2(\mathbf{y})\| \leq \frac{6}{(m(\mathbf{y}) - \|\mathbf{h}\|_\infty)^4}\|h\|_\infty.$$

Thus, the continuity of $D\mathbf{F}_2$ has been shown, and the entire proof has been completed. $\square$

Remark 5.2. The function $\mathbf{F} : U \to \ell^\infty$ is C^∞ and even C^ω, i.e. analytic. The proof of this remark is standard, but it is omitted, as we will not need differentiability beyond C^1.

Due to the control we have over the solutions, we can prove global existence and uniqueness of solutions.

Proposition 5.3. *Let* $\sup_i \lambda_i < \infty$. *Let* $\mathbf{y}_0 \in U$ *and* $\mathbf{y}(t, \mathbf{y_0})$, $t \in [0, a)$ *be the local solution to the differential equation*

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y})$$

then for all $t \in [0, a)$ *we have* $\mathbf{y}(t, \mathbf{y_0}) \in U$. *Then* $M(\mathbf{y}(t, \mathbf{y_0}))$ *and* $m(\mathbf{y}(t, \mathbf{y_0}))$ *are Lipschitz for* $t \in [0, a)$. *Furthermore, the function* $M(\mathbf{y}(t, \mathbf{y_0}))$ *is decreasing and the function* $m(\mathbf{y}(t, \mathbf{y_0}))$ *is increasing. If, in addition,* $\inf_i \lambda_i > 0$, *or, in terms of the original constants,*

$$\inf_{1 \leq i < \infty} \frac{k_i}{a_i\sqrt{\beta_i}} > 0,$$

then the words "increasing" and "decreasing" in the above statement can be replaced with "strictly increasing" and "strictly decreasing", respectively.

Proof. We claim M is Lipschitz on ℓ^∞. In fact, let $\mathbf{y}, \mathbf{z} \in \ell^\infty$. We have

$$M(\mathbf{y}) - M(\mathbf{z}) = \sup_i y_i - \sup_i z_i \leq \sup_i (y_i - z_i) = \|\mathbf{y} - \mathbf{z}\|_\infty.$$

Thus, M is Lipschitz with Lipschitz constant 1.

$M(\mathbf{y}(t, \mathbf{y_0}))$ is a composition of Lipschitz function M and differentiable function $\mathbf{y}(t, \mathbf{y}_0)$. Hence $M(\mathbf{y}(t, \mathbf{y_0}))$ is Lipschitz. Similarly, $m(\mathbf{y}(t, \mathbf{y_0}))$ is also Lipschitz.

Let $s = s(\mathbf{y}) = \sum_i \gamma_i y_i$. For each i such that $y_i \geq s$ we have the following estimate of $\dot{y}_i$:

$$\begin{aligned} \dot{y}_i &= \lambda_i\left(-\frac{1}{s^2} + \frac{1}{y_i^2}\right) = \lambda_i\left(-\frac{1}{s^2} + \frac{1}{(s + (y_i - s))^2}\right) \\ &= -\frac{2\lambda_i}{(s + \theta(y_i - s))^3}(y_i - s) \leq -\frac{2\lambda_i}{M(\mathbf{y})^3}(y_i - s). \end{aligned}$$

(We note that we used the Mean Value Theorem, and thus $\theta \in [0, 1]$.) Therefore, if $y_i \geq s$ then $\dot{y}_i \leq 0$. Also, if $y_i > s$ then $\dot{y}_i < 0$. More detailed information will be needed later, and will be derived from the above formula.

For every $\mathbf{y} \in \ell^\infty$ we define:

$$\Delta = \Delta(\mathbf{y}) = M(\mathbf{y}) - s = \sum_i \gamma_i(M(\mathbf{y}) - y_i).$$

It is clear that $\Delta \geq 0$ and $\mathbf{y}$ is a non-equilibrium iff $\Delta > 0$, as $\gamma_i > 0$ for all i. For every $\delta \in [0, \Delta]$ we also define:

$$\mathcal{I}_\delta(\mathbf{y}) = \{i \,:\, y_i \geq s + \delta\}.$$

We note that $\mathcal{I}_\Delta = \mathcal{I}$ defined in the proof of Theorem 2.1. However, now it happens that $\mathcal{I}(\mathbf{y}) = \emptyset$ when the $\sup_i y_i$ is not realized for any i, and thus the function $\mathcal{I}$ is of limited use. But for any $\delta \in (0, \Delta)$, $\mathcal{I}_\delta$ in always nonempty.

For all $i \in \mathcal{I}_\delta(\mathbf{y})$ we have the following estimate:

$$\dot{y}_i \le -\frac{2\lambda_i}{M(\mathbf{y})^3}\delta < 0.$$

We claim that if $\mathbf{z} \in B_\eta(\mathbf{y})$ then

$$\mathcal{I}_{\delta+2\eta}(\mathbf{y}) \subseteq \mathcal{I}_\delta(\mathbf{z}) \subseteq \mathcal{I}_{\delta-2\eta}(\mathbf{y}).$$

Indeed, if $z_i \ge s(\mathbf{z}) + \delta$ then $y_i \ge z_i - \eta \ge s(\mathbf{z}) + \delta - \eta \ge (s(\mathbf{y}) - \eta) + \delta - \eta = s(\mathbf{y}) + \delta - 2\eta$. Thus, one half of our claim has been shown. The other half follows from symmetry with respect to $\mathbf{y}$ and $\mathbf{z}$.

Now, let $\mathbf{y}_0 \in U$ and $\mathbf{y}_0$ be a non-equilibrium. So $\Delta(\mathbf{y}_0) > 0$. Choose positive δ and η such that $\delta + 3\eta < \Delta(\mathbf{y}_0)$. Let us study a solution $\mathbf{y}(t) = \mathbf{y}(t, \mathbf{y}_0)$ on the interval $[0, a)$ such that $\mathbf{y}(t) \in B_\eta(\mathbf{y}_0)$. We know that for all indices $i \in \mathcal{I}_\delta(\mathbf{y}(t))$ we have $\dot{y}_i(t) \le 0$. Thus, all functions $y_i(t)$ are decreasing throughout the domain $[0, a)$, as long as

$$i \in \mathcal{I}_{\delta+2\eta}(\mathbf{y}_0) \subseteq \bigcap_{\mathbf{z}\in B_\eta(\mathbf{y}_0)} \mathcal{I}_\delta(\mathbf{z})\,, \tag{5.5}$$

which is nonempty. For all other indices i we have $i \notin \mathcal{I}_{\delta+2\eta}(\mathbf{y}_0)$ and thus $y_i(0) < s(\mathbf{y}_0) + \delta + 2\eta$. Hence,

$$\begin{aligned} y_i(t) &< y_i(0) + \eta < (s(\mathbf{y}_0) + \delta + 2\eta) + \eta \\ &= s(\mathbf{y}_0) + (\delta + 3\eta) \\ &= (M(\mathbf{y}_0) - \Delta(\mathbf{y}_0)) + \delta + 3\eta. \end{aligned}$$

As long as $\delta + 3\eta < \Delta(\mathbf{y}_0)$, we have $y_i(t) < M(\mathbf{y}_0)$ for $i \notin \mathcal{I}_{\delta+2\eta}(\mathbf{y}_0)$.

Combining the estimates for both kinds of indices, we obtain $M(\mathbf{y}(t)) \le M(\mathbf{y}_0)$ for all $t \in [0, a)$.

We need to show strict monotonicity of $M(\mathbf{y}(t))$, under the assumption $\inf_i \lambda_i > 0$. (As we will see, without this assumption strict monotonicity is not even true.)

The just completed proof of ordinary monotonicity requires only one minor change to show strict monotonicity. The inequalities $\dot{y}_i \le 0$ for can be replaced with sharp inequalities for $i \in \mathcal{I}_\delta(\mathbf{y}(t))$:

$$\dot{y}_i(t) \le -\frac{2\lambda_i}{M(\mathbf{y}(t))^3}\delta \le -\frac{2\inf_i \lambda_i}{(M(\mathbf{y}_0) + \eta)^3}\delta < 0.$$

Let

$$\rho \stackrel{\text{def}}{=} \frac{2\inf_i \lambda_i}{(M(\mathbf{y}_0) + \eta)^3}\delta > 0.$$

For all indices satisfying (5.5) and all $t \in [0, a)$ we have

$$y_i(t) \le y_i(0) - \rho t.$$

The estimate for all other indices i remains the same, producing

$$M(\mathbf{y}(t)) \le \max\left(M(\mathbf{y}_0) - \rho t, M(\mathbf{y}_0) - (\Delta(\mathbf{y_0}) - \delta - 3\eta)\right).$$

Hence, as long as $\delta + 3\eta < \Delta(\mathbf{y}_0)$, we have $M(\mathbf{y}(t) < M(\mathbf{y}_0)$ for all $t \in [0, a)$. This proves strict monotonicity of $M(\mathbf{y}(t))$.

From local monotonicity proved so far we show easily that $M(y(t))$ is decreasing (or strictly decreasing) over any interval in its domain of definition. Similar arguments apply to $m(\mathbf{y}(t))$, and thus the proof of strict monotonicity on $[0, \infty)$ is complete. ☐

Lemma 5.4. *For every solution of (5.2) defined on interval $[p, q]$ and every $t \in [p, q]$, we have*

$$D^\dagger(V \circ \mathbf{y})(t) \le -\alpha(\mathbf{y}(t))(V \circ \mathbf{y})(t).$$

Proof. This is an extension of Lemma 3.11 to the infinite-dimensional case. We will make the necessary modifications in the proof of that lemma. Again, we study the function:

$$V(\mathbf{y}) = M(\mathbf{y}) - m(\mathbf{y})$$

and show

$$D^{\dagger}(V \circ \mathbf{y})(t)) \leq -\alpha(\mathbf{y}(t))(V \circ \mathbf{y})(t).$$

First, just as in the finite-dimensional case, it is true that for every pair of indices (i, j) such that $y_i \geq s \geq y_j$ we have

$$\dot{y}_i - \dot{y}_j \leq -\alpha(\mathbf{y})(y_i - y_j), \quad \text{where} \quad \alpha(\mathbf{y}) = \frac{2\inf_i \lambda_i}{M(\mathbf{y})^3}.$$

(The reader should consult Lemma 3.1.)

For any $\mathbf{y} \in U$, let $\mathcal{L}_\delta(\mathbf{y})$ be the set of all pairs (i, j) such that $y_i > s + \delta$ and $y_j < s - \delta$. Again, we have upper semi-continuity like property: if $z \in B_\eta(\mathbf{y})$ then:

$$\mathcal{L}_{\delta+2\eta}(\mathbf{y}) \subseteq \mathcal{L}_\delta(\mathbf{z}) \subseteq \mathcal{L}_{\delta-2\eta}(\mathbf{y}).$$

Let us consider the function

$$\Delta_1(\mathbf{y}) = \min\left(M(\mathbf{y}) - s(\mathbf{y}), s(\mathbf{y}) - m(\mathbf{y})\right).$$

We assume that $\Delta_1(\mathbf{y}(t)) > 0$, i.e. we are dealing with a non-equilibrium solution, and we choose $\delta > 0$ and $\eta > 0$ such that $\delta + 2\eta < \Delta_1(\mathbf{y})$ and $\delta - 2\eta > 0$. Let $\epsilon > 0$ be chosen so that for all $t' \in (t - \epsilon, t + \epsilon) \cap [0, a)$ there is $\mathbf{y}(t') \in B_\eta(\mathbf{y}_0)$. Thus,

$$V(\mathbf{y}(t')) = \sup_{(i,j)\in\mathcal{L}_\delta(\mathbf{y}(t'))} (y_i(t') - y_j(t')) = \sup_{(i,j)\in\mathcal{L}_{\delta-2\eta}(\mathbf{y}(t))} (y_i(t') - y_j(t')).$$

We complete the proof in the same fashion as the finite-dimensional case, except we use $\mathcal{L} = \mathcal{L}_{\delta-2\eta}(\mathbf{y}(t))$ instead of $\mathcal{L}(\mathbf{y}(t))$ used there. ☐

The next theorem is analogous to Theorem 3.13, part (a) and (b). The remaining parts (c) and (d) remain true, with almost identical proofs. We included only the parts (a) and (b), as they require the most substantial modification of the proof, consisting in replacing the results of Section 2, which relies upon the finite dimensionality of the phase space, with an argument based on a Cauchy condition in a Banach space.

Theorem 5.5. *(a) For every C the set U_C is forward invariant. The system (5.1) is forward complete, and thus, the function $\mathbf{F}$ is an infinitesimal generator of a one-parameter semi-group of transformations (see Appendix D).*
(b) Under the assumption $\inf_i \lambda_i > 0$, there exists a unique equilibrium $\bar{\mathbf{y}}$ such that for all $t \in [0, \infty)$:

$$\|\mathbf{y}(t, \mathbf{y}_0) - \bar{\mathbf{y}}\|_\infty \leq (M(\mathbf{y}_0) - m(\mathbf{y}_0)) \exp(-\alpha_0 t).$$

where

$$\alpha_0 = \frac{2\inf_i \lambda_i}{M(\mathbf{y}_0)^3}.$$

Proof. (a) Since $M(\mathbf{y})$ is decreasing and $m(\mathbf{y})$ is increasing, for every C the set U_C is forward invariant. As in the proof of Lemma 5.1, $\|\mathbf{F}(\mathbf{y})\| \leq 2C^2 \sup_i \lambda_i$ and $\|D\mathbf{F}(\mathbf{y})\| \leq 4C^3 \sup_i \lambda_i$ on U_C. Thus for any initial condition $\mathbf{y}_0 \in U$, we choose $C > \max(1/m(\mathbf{y}_0), M(\mathbf{y}_0))$ such that $\mathbf{y}_0 \in U_C$. By Appendix C, Lemma C.1, the system is forward complete, as we easily verify that the l^∞-distance from U_C to the complement of U is $\inf_{U_C} m > 1/C$. In fact, we estimate the left-hand side of inequality (C.1), with $r \nearrow 1/C$, not to exceed $4C^3 \sup_i \lambda_i + 2C^2 \sup_i \lambda_i \cdot C = 6C^3 \sup_i \lambda_i$.
(b) For any $\mathbf{y}_0 \in U$, we have $M(\mathbf{y}(t)) \leq M(\mathbf{y}_0)$ for $t \geq 0$. Then

$$-\alpha(\mathbf{y}(\mathbf{t})) \leq -\alpha_0 = -\frac{2\inf_i \lambda_i}{M(\mathbf{y}_0)^3}$$

for all $t \geq 0$. From Lemma 5.4, we have

$$D^{\dagger}(V(\mathbf{y}(t)) \leq -\alpha_0 V(\mathbf{y}(t)).$$

$V(\mathbf{y}(t)) = M(\mathbf{y}(t)) - m(\mathbf{y}(t))$ is continuous because $M(\mathbf{y}(t))$ and $m(\mathbf{y}(t))$ are Lipschitz. From Lemma 3.9, we have

$$V(\mathbf{y}(t)) \leq V(\mathbf{y}_0)\exp(-\alpha_0 t)\,.$$

For $\bar{m} \leq \bar{M}$, we define the set

$$P(\bar{m}, \bar{M}) = \left\{\mathbf{y} \in \ell^\infty :\ \bar{m} \leq m(\mathbf{y}) \leq M(\mathbf{y}) \leq \bar{M}\right\}.$$

Then $P(\bar{m}, \bar{M})$ is a closed set in ℓ^∞ and its diameter is $\bar{M} - \bar{m}$. Since $\inf_i \lambda_i > 0$, we have $\alpha_0 > 0$. Thus for every $\epsilon > 0$ there exists $T_0 > 0$ such that $V(\mathbf{y}(T_0)) < \epsilon$. Since $M(\mathbf{y})$ is decreasing and $m(\mathbf{y})$ is increasing, the solution $\mathbf{y}(t)$ (when $t \geq T_0$) is within the closed set $P(m(\mathbf{y}(T_0)), M(\mathbf{y}(T_0))$ whose diameter is $V(\mathbf{y}(T_0)) < \epsilon$. Thus the solution $\mathbf{y}(t)$ satisfies the Cauchy condition, and as such it converges to a limit $\bar{\mathbf{y}}$ because ℓ^∞ is complete. Also,

$$\bar{\mathbf{y}} \in \bigcap_{t \geq 0} P(m(\mathbf{y}(t)), M(\mathbf{y}(t))\,.$$

Since $M(\mathbf{y}(t)) - m(\mathbf{y}(t)) = V(\mathbf{y}(t)) \to 0$ as $t \to \infty$, we have $M(\bar{\mathbf{y}}) = m(\bar{\mathbf{y}})$. Thus $\bar{\mathbf{y}}$ is an equilibrium. Also,

$$\mathbf{y}(t),\ \bar{\mathbf{y}} \in P(m(\mathbf{y}(t)), M(\mathbf{y}(t))\,.$$

Therefore,

$$\|\mathbf{y}(t) - \bar{\mathbf{y}}\| \leq V(\mathbf{y}(t)) \leq V(\mathbf{y}_0)\exp(-\alpha_0 t) = (M(\mathbf{y}_0) - m(\mathbf{y}_0))\exp(-\alpha_0 t)\,.$$

□

As we have seen, in order to prove strict monotonicity we had to assume that

$$\inf_i \lambda_i > 0,$$

which makes the inequality $\dot{y}_i < 0$ uniform in i. Otherwise, the function M can be constant along some non-equilibrium trajectories. This indeed happens as our next theorem shows. Before we formulate the theorem, let us introduce the following notation. For every $\mathbf{y} \in \ell^\infty$, let $L(\mathbf{y})$ be the set of all limit points of the sequence $\mathbf{y}$, i.e. the set of all numbers $z \in \mathbb{R}$ such that there is a subsequence $i_k \nearrow \infty$ for which $\lim_{k\to\infty} y_{i_k} = z$.

Theorem 5.6. *Let us assume that* $\lim_{i\to\infty} \lambda_i = 0$. *Let* $\mathbf{y}_0$ *be a positive sequence in* ℓ^∞ *such that* $\mathbf{y}_0 \in U_C$ *for some* $C > 0$. *Let* $\mathbf{y}(t, \mathbf{y}_0)$ *be the solution of the system (2.1) defined on* $[0, \infty)$. *Then for all* $t \geq 0$

$$L(\mathbf{y}(t)) = L(\mathbf{y_0}).$$

Proof. $\mathbf{y}(t) \in U_C$ for all $t \geq 0$ due to the forward invariance of U_C. The modulus of the derivative $|\dot{y}_i|$ satisfies the inequality:

$$|\dot{y}_i| = \left|\lambda_i\left(-\frac{1}{s^2} + \frac{1}{y_i^2}\right)\right| \leq \frac{\lambda_i}{m(\mathbf{y})} \leq C^2\lambda_i$$

and thus as i goes to infinity, the slope of $y_i(t)$ uniformly converges to 0. This implies that over any fixed domain $[0, a]$ the functions $y_i(t)$ vary less and less as $i \to \infty$, and thus for all t the accumulation points of the sequence $y_i(t)$ are the same. More precisely, for each i we have

$$|y_i(t) - y_i(0)| = |\dot{y}_i(t_i')| \leq C^2\lambda_i,$$

where $t_i' \in (0, t)$. Hence, for every subsequence $i_k \nearrow \infty$ we have $\lim_{k\to\infty} y_{i_k}(t) = \lim_{k\to\infty} y_{i_k}(0)$. in the sense that either both limits exist and are equal, or they both do not exist. Thus, the limit points of the sequences $y_i(t)$ and $y_i(0)$ are identical. □

Example 5.7. *If* $\lim_{i\to\infty}\lambda_i = 0$*, the above theorem gives rise to a family of examples for which the solutions of the system do not go to equilibria, but instead oscillate wildly. Let* $K \subset (0,\infty)$ *be any set of more than 1 point and let us consider the set of initial conditions* $\mathbf{y}_0$ *for which* $L(\mathbf{y}_0) = K$*. For any such set, the solution does not go to an equilibrium. We note that* K *could be an interval or a* Cantor set. *For instance,* $\mathbf{y}_0$ *can be chosen as a sequence of all the rational numbers in* $[1,2]$.

The following theorem summarizes the results of this section.

Theorem 5.8. *The solution to model (5.1) is forward complete for any initial condition* $\mathbf{y}_0$ *if the following conditions are satisfied:*

(1) $\mathbf{y}_0 \in l^\infty$;
(2) $\sup_{1\le i<\infty} \frac{k_i}{a_i\sqrt{\beta_i}} < \infty$;
(3) $\inf_i y_i > 0$.

Moreover, if

(1) $\sum_{i=1}^{\infty} \sqrt{\beta_i} = \sqrt{b}$,
(2) $\inf_{1\le i<\infty} \frac{k_i}{a_i\sqrt{\beta_i}} > 0$,

then there is a unique set of equilibria of model (5.1) which is a ray emanating from the origin and is a globally strongly attracting invariant set.

Appendix A. The Existence and Uniqueness Theorem

For the convenience of the reader we state the existence and uniqueness theorems for differential equations on Banach spaces. In this appendix, $B_r(\mathbf{x})$ denotes the ball of radius r about $\mathbf{x}$:

$$B_r(\mathbf{x}) = \{\mathbf{y} \in X \,:\, \|\mathbf{y} - \mathbf{x}\| < r\}.$$

Theorem A.1. *Let* X *be a Banach space,* $A > 0$ *and* $\mathbf{F} : B_r(\mathbf{x}_0) \times [t_0, t_0 + A) \to X$ *be a function satisfying the Lipschitz condition with constant* $L \in \mathbb{R}$ *with respect to the first variable, i.e. for every* $\mathbf{x}_1, \mathbf{x}_2 \in B_r(\mathbf{x}_0)$ *and* $t \in [t_0, t_0 + A)$*:*

$$\|\mathbf{F}(\mathbf{x}_1, t) - \mathbf{F}(\mathbf{x}_2, t)\| \le L\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

Moreover, let us assume that $\mathbf{F}$ *is continuous in the second argument. Then there is a unique solution* $\mathbf{x} : [t_0, t_0 + a) \to X$ *to the initial value problem:*

$$\left\{\begin{array}{rcl} \dfrac{d\mathbf{x}}{dx} & = & \mathbf{F}(\mathbf{x}, t), \\ \mathbf{x}(t_0) & = & \mathbf{x}_0, \end{array}\right. \tag{A.1}$$

where a *is any number* $\le A$ *such that:*

$$a \le \left(L + \frac{1}{r} \sup_{t\in[t_0,t_0+A)} \|\mathbf{F}(\mathbf{x}_0, t)\| \right)^{-1}. \tag{A.2}$$

Proof. This argument is standard, and thus we only sketch it, with the main goal of extracting the estimate of a. Let $C^0([t_0, t_0 + a), X)$ be the space of continuous functions from $[t_0, t_0 + a)$ to X with the sup-norm:

$$\|\mathbf{x}\| = \sup_{t\in[t_0,t_0+a)} \|\mathbf{x}(t)\|.$$

Let Y be the subset consisting of functions $\mathbf{x} : [t_0, t_0 + a) \to X$ which satisfy the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ and such that $\mathbf{x}([t_0, t_0 + a)) \subseteq B_r(\mathbf{x}_0)$. Let $T : Y \to C^0([t_0, t_0 + a), X)$ be the integral operator defined by:

$$(T\mathbf{x})(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{F}(\mathbf{x}(\tau), \tau)\, d\tau.$$

We need to show that $T(Y) \subseteq Y$ and that T is a contraction. First, for every $\tau \in [t_0, t_0 + a)$ we have $\|\mathbf{F}(\mathbf{x}(\tau), \tau) - \mathbf{F}(\mathbf{x}_0, \tau)\| \le L\|\mathbf{x}(\tau) - \mathbf{x}_0\| \le Lr$. Thus,

$$\begin{aligned}
\|(T\mathbf{x})(t) - \mathbf{x}_0\| &= \left\| \int_{t_0}^{t} \mathbf{F}(\mathbf{x}(\tau), \tau)\, d\tau \right\| \le \int_{t_0}^{t} \|\mathbf{F}(\mathbf{x}(\tau), \tau)\|\, d\tau \\
&\le \int_{t_0}^{t} (\|\mathbf{F}(\mathbf{x}_0, \tau)\| + Lr)\, d\tau \le a \left(\sup_{\tau \in [t_0, t_0 + A)} \mathbf{F}(\mathbf{x}_0, \tau) + Lr \right).
\end{aligned}$$

By definition of a, this number is smaller than r and $\mathbf{x}([t_0, t_0 + a)) \subseteq B_r(\mathbf{x}_0)$. Thus, $T(Y) \subseteq Y$.

Let us show that T is a contraction. If $\mathbf{x}_1, \mathbf{x}_2 \in Y$ then

$$\begin{aligned}
\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| &= \left\| \int_{t_0}^{t} (\mathbf{F}(\mathbf{x}_1(\tau), \tau) - \mathbf{F}(\mathbf{x}_2(\tau), \tau))\, d\tau \right\| \\
&\le \int_{t_0}^{t} \|\mathbf{F}(\mathbf{x}_1(\tau), \tau) - \mathbf{F}(\mathbf{x}_2(\tau), \tau)\|\, d\tau \\
&\le a \sup_{\tau \in [t_0, t_0 + a)} \|\mathbf{F}(\mathbf{x}_1(\tau), \tau) - \mathbf{F}(\mathbf{x}_2(\tau), \tau)\| \\
&\le a \sup_{\tau \in [t_0, t_0 + a)} L\, \|\mathbf{x}_1(\tau) - \mathbf{x}_2(\tau)\| \\
&= aL\|\mathbf{x}_1 - \mathbf{x}_2\|.
\end{aligned}$$

The definition of a implies that $aL \le 1$, and thus $T : Y \to Y$ is a weak contraction. Moreover, for every a such that $aL < 1$ the Banach Contraction Principle implies both the existence and uniqueness of the solution. Since this works for every a such that $aL < 1$, we can also deduce the existence and uniqueness for $aL = 1$ as well. □

We note that the Lipschitz condition for $\mathbf{F}$ of class C^1 is verified by means of finding the norms of the derivatives, which is justified by the Mean Value Theorem:

$$L = \sup_{(\mathbf{x}, t) \in B_r(\mathbf{x}_0) \times [t_0, t_0 + A)} \|D_{\mathbf{x}} \mathbf{F}(\mathbf{x}, t)\|$$

where the notation $D_{\mathbf{x}}$ is used for the "partial" over $\mathbf{x}$ only.

Appendix B. The flow of an autonomous system

An initial value problem for an autonomous system can be stated as follows:

$$\text{(B.1)} \qquad \begin{cases} \dfrac{d\mathbf{x}}{dx} &= \mathbf{F}(\mathbf{x}), \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{cases}$$

where $\mathbf{F} : U \to X$ is a C^1 function defined on an open subset U of a Banach space X.

For every initial condition $\mathbf{x}_0 \in U$ there is a solution $\mathbf{x} : (a, b) \to X$ such that $a < 0$ is the smallest possible, $b > 0$ is the largest possible and $\mathbf{x}(0) = \mathbf{x}_0$. Clearly, a and b may depend on $\mathbf{x}_0$. Let us define functions $\tau_\pm : U \to \mathbb{R}^+$ by the formulas $\tau_-(\mathbf{x}_0) = a$ and $\tau_+(\mathbf{x}_0) = b$. Theorem A.1 implies that τ_+ is and upper semi-continuous function and τ_- is a lower semi-continuous function. Let us consider the set $D \subseteq U \times \mathbb{R}$:

$$D = \{(\mathbf{x}, t) \in D \subseteq U \times \mathbb{R} \;:\; \tau_-(\mathbf{x}) < t < \tau_+(\mathbf{x})\}.$$

The semi-continuity of $\tau_\pm$ implies that D is an open subset of $U \times \mathbb{R}$.

Definition B.1. *The* flow *of the system is the map $\varphi : D \to U$ defined by the condition that for every fixed $\mathbf{x}_0$ the function $\varphi(\mathbf{x}_0, t) = \mathbf{x}(t)$. We often write $\varphi^t(\mathbf{x}_0)$ in place of $\varphi(\mathbf{x}_0, t)$. The map φ^t is only well defined on the open set $\{\mathbf{y} \in U \;:\; \tau_-(\mathbf{y}) < t < \tau_+(\mathbf{y})\}$.*

Definition B.2. *The* trajectory *of* $\mathbf{x} \in U$ *is the function*

$$(-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \ni t \mapsto \varphi(\mathbf{x}, t) \in U.$$

Thus, the trajectory may be regarded as a path in U*. Similarly, the* forward *and* backward *trajectory are defined by restricting* t *to* $(-\tau_-(\mathbf{x}), 0]$ *and* $[0, \tau_+(\mathbf{x}))$*, respectively.*

Definition B.3. *The* orbit $\mathscr{O}(\mathbf{x})$ *of* $\mathbf{x} \in U$ *is the set which is the image of the corresponding trajectory. More precisely,*

$$\mathscr{O}(\mathbf{x}) = \{\varphi(\mathbf{x}, t) \, : \, t \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))\}.$$

Similarly, the forward orbit $\mathscr{O}^+(\mathbf{x})$ *and* backward orbit $\mathscr{O}^-(\mathbf{x})$ *are defined as follows:*

$$\begin{aligned} \mathscr{O}^+(\mathbf{x}) &= \{\varphi(\mathbf{x}, t) \, : \, t \in [0, \tau_+(\mathbf{x}))\}, \\ \mathscr{O}^-(\mathbf{x}) &= \{\varphi(\mathbf{x}, t) \, : \, t \in (-\tau_-(\mathbf{x}), 0]\}. \end{aligned}$$

Definition B.4. *The* ω-limit set $\omega(\mathbf{x})$ *is defined as follows:*

$$\omega(\mathbf{x}) = \bigcap_{t \geq 0} \overline{\mathscr{O}^+(\varphi^t(\mathbf{x}))}.$$

Similarly, the α-limit set $\alpha(\mathbf{x})$ *is defined as follows:*

$$\alpha(\mathbf{x}) = \bigcap_{t \leq 0} \overline{\mathscr{O}^-(\varphi^t(\mathbf{x}))}.$$

Remark B.1. The families of sets in the above definition are decreasing. For instance, if $t_1, t_2 \in [0, \tau_+(\mathbf{x}))$ and $t_1 \leq t_2$ then $\mathscr{O}^+\left(\varphi^{t_1}(\mathbf{x})\right) \supseteq \mathscr{O}^+\left(\varphi^{t_2}(\mathbf{x})\right)$. Therefore, $\overline{\mathscr{O}^+(\varphi^{t_1}(\mathbf{x}))} \supseteq \overline{\mathscr{O}^+(\varphi^{t_2}(\mathbf{x}))}$. If $\overline{\mathscr{O}^+(\mathbf{x})}$ is compact then also automatically the intersection is compact and non-empty. If the containing Banach space is finite-dimensional then the boundedness of $\mathscr{O}^+(\mathbf{x})$ implies compactness of $\overline{\mathscr{O}^+(\mathbf{x})}$, and $\omega(\mathbf{x}) \neq \emptyset$. In the infinite-dimensional case, the compactness does not follow from boundedness, and proving that $\omega(\mathbf{x}) \neq \emptyset$ may be considerably more difficult. [1]

Appendix C. Completeness of an autonomous system

Let us begin with the following definition.

Definition C.1. *The initial condition* $\mathbf{x}_0$ *in the initial value problem (B.1) is called* forward complete (backward complete) *if* $\tau_+(\mathbf{x}_0) = \infty$ $(\tau_-(\mathbf{x}_0) = -\infty)$*. equivalently, that the solution passing through* $\mathbf{x}_0$ *extends to* $[0, \infty)$ $((-\infty, 0])$*. This initial condition is called* complete *if it is both forward and backward complete.*

The system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ *and the corresponding flow* φ *are called* forward complete (backward complete, complete)*, if every* $\mathbf{x}_0$ *is a forward complete (backward complete, complete) initial condition, respectively.*

We will formulate sufficient conditions of forward completeness.

For any point $x \in X$, let us consider the distance from the boundary of U:

$$\operatorname{dist}(\mathbf{x}, X \backslash U) = \inf_{\mathbf{y} \notin U} \|\mathbf{x} - \mathbf{y}\|.$$

(The right-hand side is ∞ if $U = X$.) This is also the radius of the maximal ball $B_r(\mathbf{x}_0)$ contained in U. It is known, and easy to check using the triangle inequality, that $\operatorname{dist}(\mathbf{x}, X \backslash U)$ is a continuous, and even Lipschitz with constant 1, function of $\mathbf{x}$.

[1] This difficulty is clear in this paper. We note that there is no analogue of Theorem 2.1 in Section 5. Only after proving a stronger, exponential estimate in Theorem 5.5, we were able to derive the conclusions of Theorem 2.1.

Lemma C.1. *Let $\mathbf{F} : U \to X$ of class C^1 and let $\mathbf{x} : [0, b) \to U$ be a maximal forward solution, i.e. defined on a maximal interval $[0, b)$, to the initial value problem (B.1). The sufficient condition for $b = \infty$ is:*

$$\text{(C.1)} \qquad \sup_{t\in[0,b)} \left(\inf_{r\in(0,\mathrm{dist}(\mathbf{x}(t),X\setminus U))} \left(\sup_{\mathbf{y}\in B_r(\mathbf{x}(t))} \|D\mathbf{F}(\mathbf{y})\| + \frac{1}{r}\|\mathbf{F}(\mathbf{x}(\mathbf{t}))\| \right) \right) < \infty.$$

Proof. By Theorem A.1, for every $t_0 \in [0, b)$ the solution $\mathbf{x}(t)$ is defined on at least the interval $[t_0, t_0+a)$ where a is the inverse of the left-hand-side of inequality (C.1). If $b < \infty$ and $t_0 \in (b-a, b)$, we obtain a contradiction with the maximality of b. □

Corollary C.2. *Let $\mathbf{F} : U \to X$ be of class C^1. A sufficient condition of completeness of the system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ is that both of the following inequalities are satisfied:*

(1) $\sup_{\mathbf{x}\in U} \|D\mathbf{F}(\mathbf{x})\| < \infty$;
(2) $\sup_{\mathbf{x}\in U} \dfrac{\|\mathbf{F}(\mathbf{x})\|}{\mathrm{dist}(\mathbf{x}, X\setminus U)} < \infty$.

Note: condition (2) is automatically satisfied if $U = X$.

Lemma C.3. *Let $\mathbf{F} : U \to X$ be of class C^1 and let $\mathbf{x} : [0, b) \to U$ be a maximal forward solution to the initial value problem (B.1). Let us assume that there is a compact set $K \subset U$ such that $\mathbf{x}([0, b)) \subseteq K$. Then $b = \infty$.*

Proof. Let $L_0 = \sup_{\mathbf{x}\in K} \|D\mathbf{F}(\mathbf{x})\|$ and $C = \sup_{\mathbf{x}\in K} \|\mathbf{F}(\mathbf{x})\|$. Both constants are finite as suprema of continuous functions over a compact set. Let $L = L_0 + 1$ and $V = \{\mathbf{x} \in U : \|D\mathbf{F}(\mathbf{x})\| < L\}$. We note that V is open and contains K. Let $r_0 = \mathrm{dist}(K, X\setminus V)$. Clearly, $r_0 > 0$, as it is an infimum of a continuous function $\mathrm{dist}(\mathbf{x}, X\setminus V)$ over a compact set. It is easy to see that the left-hand side of inequality (C.1) is not greater than $L + (1/r_0)C$, using fixed $r = r_0$ for all t. Thus, Lemma C.1 implies the current lemma. □

Appendix D. Flows, semi-flows and semi-groups of transformations

If the system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$, is forward complete on U (see Appendix C) then we may define *a one-parameter semi-group of transformations* associated with this system by restricting φ to $U \times [0, \infty)$. By definition, the one-parameter semi-group of transformations is a family of mappings $\varphi^t : U \to U$, $t \geq 0$, whose domains are now unrestricted.

One-parameter semi-groups of transformations are defined by the following *semi-group property*:

(1) $\varphi^0 = id_U$;
(2) For all $s, t \in [0, \infty)$ we have $\varphi^s \circ \varphi^t = \varphi^{s+t}$.

If the system is complete then $\varphi : U \times \mathbb{R} \to U$ and $\varphi^t : U \to U$. Moreover, the family $(\varphi^t)_{t\in\mathbb{R}}$ is a *one-parameter group of transformations*. We note that the inverse of the map φ^t is φ^{-t}.

Thus, the terms "complete semi-flow" and "one-parameter semi-group of transformations" will be considered synonymous. Also, the terms "complete flow" and "one-parameter group of transformations" are considered synonymous.

When the semi-group or group of transformations comes from the differential equation $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$, the map $\mathbf{F} : U \to X$ is called the *infinitesimal generator* (or simply *the generator*) of the semi-group or group.

Appendix E. An extension of LaSalle's Invariance Principle

In this appendix we prove a theorem which generalized the well-known LaSalle's Invariance Principle [4, 5].

Definition E.1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $g : I \to \mathbb{R}$ is called* quasi-strictly decreasing *if*

1. *For every $t_1, t_2 \in I$ such that $t_1 \geq t_2$ we have $g(t_2) \leq g(t_1)$, i.e. g is decreasing;*
2. *For every $t_1 \in I$ and t_1 is not the right endpoint of I, there exists $t_2 \in I$ such that $g(t_2) < g(t_1)$, i.e. g has no global minimum on I.*

The following definition of forward invariance does not assume that the system is complete:

Definition E.2. *A subset $S \subseteq U$ is called* forward invariant *for the differential equation $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ defined on U if for every $\mathbf{x} \in S$ we have $\mathscr{O}^+(\mathbf{x}) \subseteq S$.*

Theorem E.1. *Let $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ be a differential equation on a Banach space X, where $\mathbf{F} : U \to X$ is of class C^1 and U is an open subset of X. Let $S \subseteq X$ be a forward invariant set with respect to this system. Also, we assume that*

1. *There exists a continuous function V from X to $\mathbb{R}$, such that: for every $\mathbf{x} \notin S$ the function $t \mapsto V(\varphi(\mathbf{x}, t))$ is quasi-strictly decreasing on $[0, \tau_+(\mathbf{x}))$;*
2. *For every $\mathbf{x} \in U$ the closure of the forward orbit $\overline{\mathscr{O}^+(\mathbf{x})}$ is a compact set.*

Then the system is forward complete and every trajectory is attracted to S. Furthermore, if $V|_S$ is injective then S is globally strongly attracting, i.e. every trajectory outside S converges to a fixed point in S.

Proof. Let us fix any initial condition $\mathbf{x_0}$. The forward completeness of a solution with any initial condition is immediate from Lemma C.3. The ω-limit set $\omega(\mathbf{x}_0)$ is non-empty because the forward orbit $\mathscr{O}^+(\mathbf{x})$ lies inside of a compact set. We claim that V is invariant on $\omega(\mathbf{x}_0)$ (Invariance Principle). Let $\mathbf{y}_1 \neq \mathbf{y}_2$ and $\mathbf{y}_1, \mathbf{y}_2 \in \omega(\mathbf{x}_0)$. Then there is a strictly increasing time sequence $(t_n)_{n=1}^{\infty}$, such that $\lim_{n\to\infty} t_n = \infty$ and for $l = 1, 2, \ldots$ we have:

1. $\varphi^{t_{2l-1}}(\mathbf{x}_0) = \mathbf{y}_1$;
2. $\varphi^{t_{2l}}(\mathbf{x}_0) = \mathbf{y}_2$.

Since $V(\varphi^t(\mathbf{x}_0))$ is continuous and decreasing,

$$
\begin{aligned}
V(\mathbf{y}_1) &= \lim_{l\to\infty} V(\varphi^{t_{2l-1}}(\mathbf{x}_0)) \geq \lim_{l\to\infty} V(\varphi^{t_{2l}}(\mathbf{x}_0)) = V(\mathbf{y}_2),\\
V(\mathbf{y}_1) &= \lim_{l\to\infty} V(\varphi^{t_{2l+1}}(\mathbf{x}_0)) \leq \lim_{l\to\infty} V(\varphi^{t_{2l}}(\mathbf{x}_0)) = V(\mathbf{y}_2).
\end{aligned}
$$

This implies $V(\mathbf{y}_1) = V(\mathbf{y}_2)$.

We next claim that $\omega(\mathbf{x}_0) \subseteq S$. Otherwise, if $\mathbf{y} \in \omega(\mathbf{x}_0)$ and $\mathbf{y} \notin S$, by hypothesis, $V(\varphi^t(\mathbf{y}))$ is quasi-strictly decreasing and is not constant for all t. On the other hand, $\omega(\mathbf{x}_0)$ is invariant and trajectory $\varphi^t(\mathbf{y})$ lies in $\omega(\mathbf{x}_0)$ so that $V(\varphi^t(\mathbf{y}))$ is constant for all t. Contradiction.

Hence, S is attracting since $\varphi^t(\mathbf{x}_0)$ is arbitrarily chosen.

If V is injective on S, then for all $\mathbf{x}_0 \notin S$ the set $\omega(\mathbf{x}_0)$ consists of a single point, which must be a fixed point in S. Therefore, S is strongly attracting. □

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